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轨道动力学中的数值积分方法

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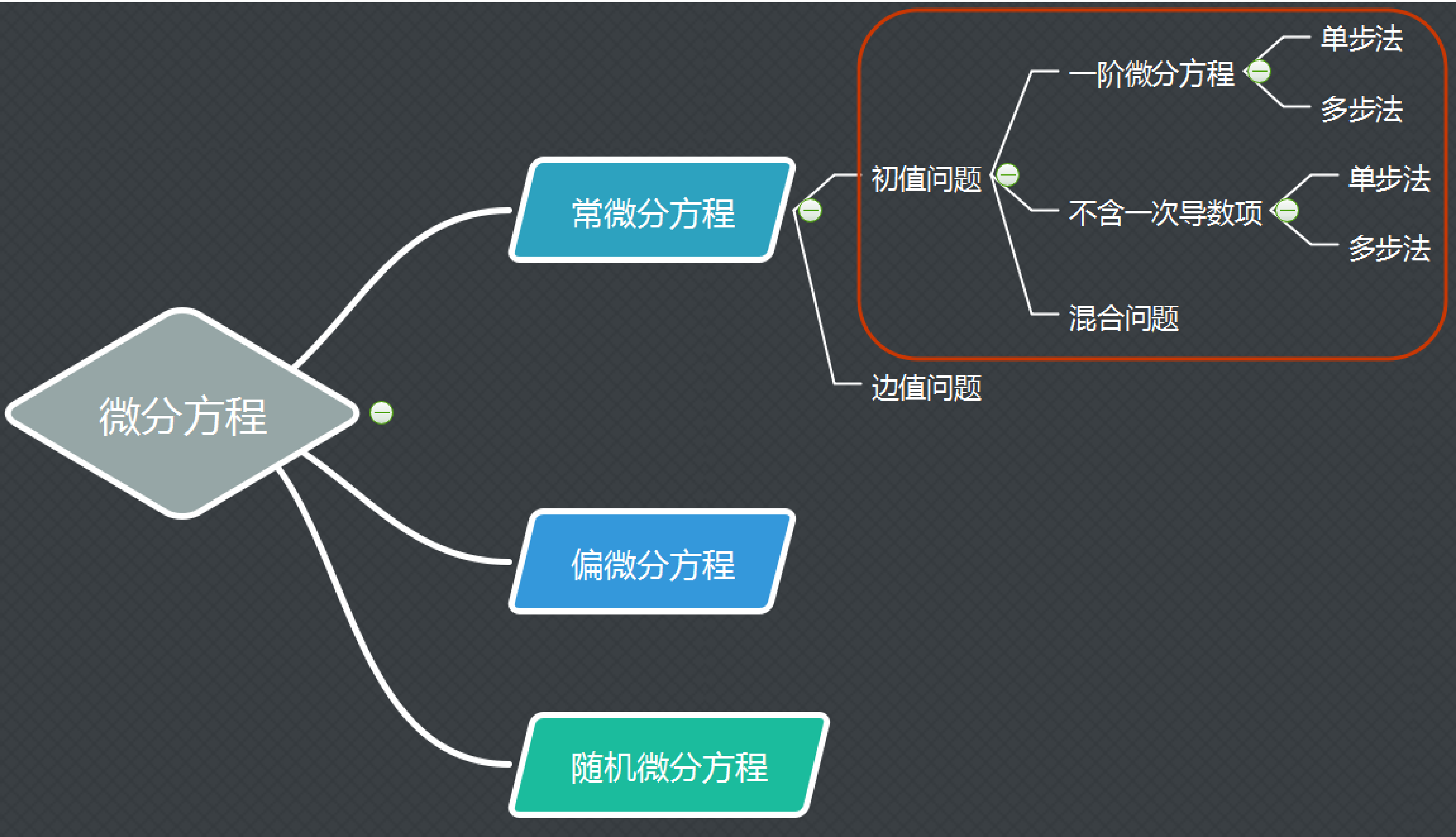
主要内容

- 单步法
- 多步法
- 辛积分器
- 精细积分法

主要内容



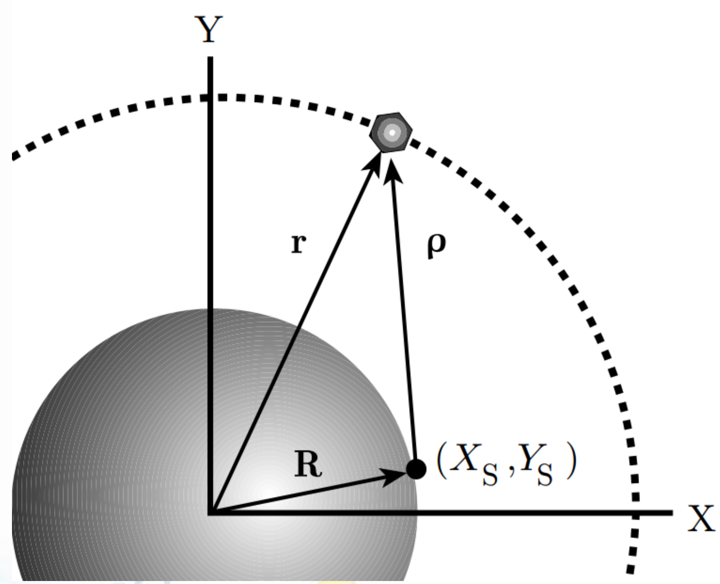
微分方程
常微分方程
初值问题
一阶微分方程
单步法
多步法
不含一次导数项
单步法
多步法
混合问题
边值问题
偏微分方程
随机微分方程

卫星动力学方程初值问题（二维范例）

$$\dot{\mathbf{X}} = F(\mathbf{X}, t)$$

$$\boldsymbol{X}(t_0) = \boldsymbol{X}_0$$

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ U \\ V \\ \mu \\ X_S \\ Y_S \end{bmatrix}$$



$$\begin{aligned}\dot{\mathbf{X}}(t) &= F(\mathbf{X}, t) = F(\mathbf{X}^*, t) + \left[\frac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^* [\mathbf{X}(t) - \mathbf{X}^*(t)] \\ &+ O_F [\mathbf{X}(t) - \mathbf{X}^*(t)]\end{aligned}$$

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$$

$$A(t) = \left[\frac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^*$$

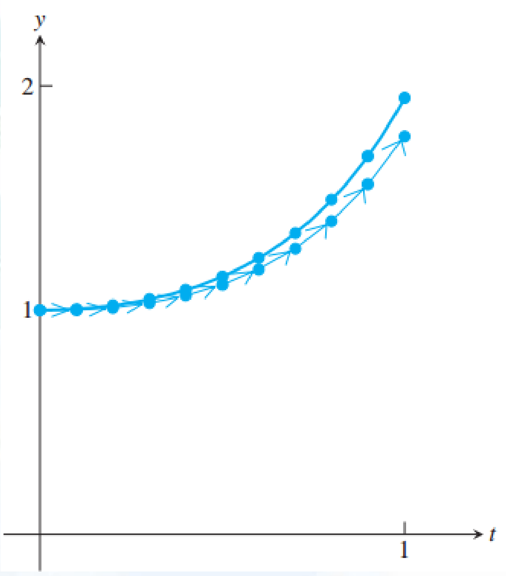
$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{U} \\ \dot{V} \\ \dot{\mu} \\ \dot{X}_S \\ \dot{Y}_S \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} U \\ V \\ -\frac{\mu X}{r^3} \\ -\frac{\mu Y}{r^3} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial F(\mathbf{X}^*, t)}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial U} & \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial \mu} & \frac{\partial F_1}{\partial X_S} & \frac{\partial F_1}{\partial Y_S} \\ \frac{\partial F_2}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_2}{\partial Y_S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_7}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_7}{\partial Y_S} \end{bmatrix}^*$$

欧拉折线的收敛性

$$y' = f(x, y), \qquad y(x_0) = y_0$$

$$\begin{aligned} y_1 - y_0 &= (x_1 - x_0) f(x_0, y_0) \\ y_2 - y_1 &= (x_2 - x_1) f(x_1, y_1) \\ &\cdots \\ y_n - y_{n-1} &= (x_n - x_{n-1}) f(x_{n-1}, y_{n-1}) \end{aligned}$$



$$|f(x, z) - f(x, y)| \le L|z - y|$$

Let $f(x, y)$ be continuous, and $|f|$ be bounded by A and satisfy the Lipschitz condition on

$$D = \Big\{(x, y) \mid x_0 \le x \le X,\ |y - y_0| \le b\Big\}.$$

If $X - x_0 \le b/A$, then we have:

a) *For $|h| \to 0$ the Euler polygons $y_h(x)$ converge uniformly to a continuous function $\varphi(x)$.*

b) *$\varphi(x)$ is continuously differentiable and solution of (7.1)* $x_0 \le x \le X$.

c) *There exists no other solution of* on $x_0 \le x \le X$.

欧拉中点格式

$$y' = f(x, y), \qquad y(x_0) = y_0$$

$$y(x_0 + h_0) \approx y_1 = y_0 + h_0 f\left(x_0 + \frac{h_0}{2}\right)$$

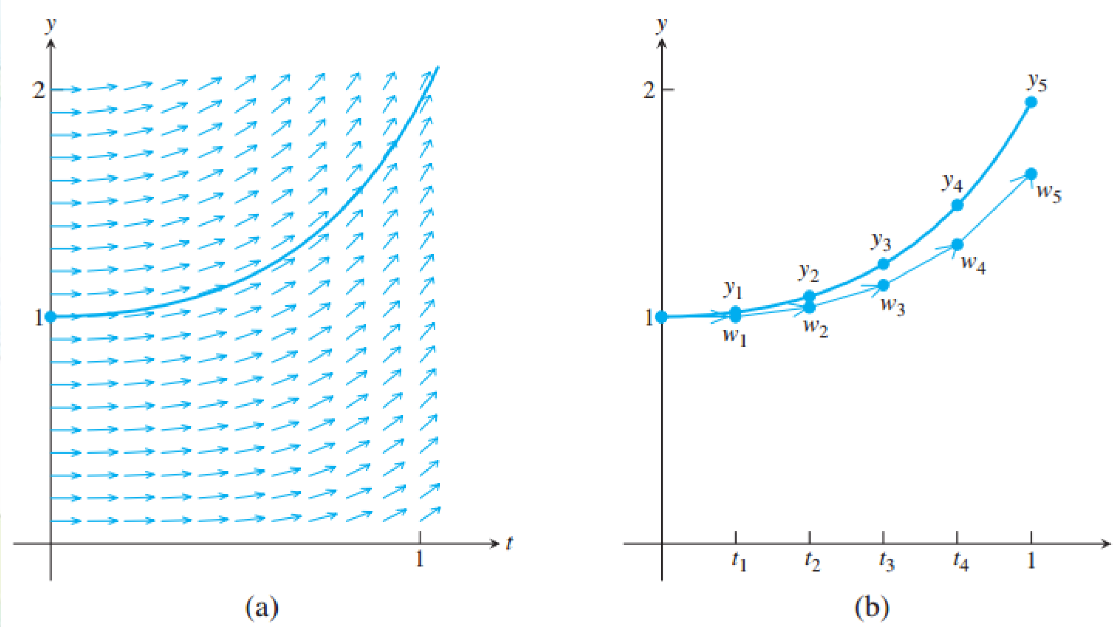
$$y(x_1 + h_1) \approx y_2 = y_1 + h_1 f\left(x_1 + \frac{h_1}{2}\right)$$

$$\cdots$$

$$k_1 = f(x_0, y_0)$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right)$$

$$y_1 = y_0 + hk_2.$$



欧拉中点格式

$$
\begin{aligned}
y_1 &= y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f_0\right) \\
&= y_0 + hf(x_0, y_0) + \frac{h^2}{2}\left(f_x + f_y f\right)(x_0, y_0) \\
&\qquad + \frac{h^3}{8}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2\right)(x_0, y_0) + \ldots.
\end{aligned}
$$

$$
\begin{aligned}
y(x_0 + h) &= y_0 + hf(x_0, y_0) + \frac{h^2}{2}\left(f_x + f_y f)(x_0, y_0\right) \\
&\qquad + \frac{h^3}{6}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_y f_x + f_y^2 f\right)(x_0, y_0) + \ldots.
\end{aligned}
$$

$$
y(x_0 + h) - y_1 = \frac{h^3}{24}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2 + 4(f_y f_x + f_y^2 f)\right)(x_0, y_0) + \ldots.
$$

一般显式RK方法

Definition 1.1. Let s be an integer (the "number of stages") and $a_{21}, a_{31}, a_{32}, \ldots,$ $a_{s1}, a_{s2}, \ldots, a_{s,s-1}, b_1, \ldots, b_s, c_2, \ldots, c_s$ be real coefficients. Then the method

$$
\begin{aligned}
k_1 &= f(x_0, y_0) \\
k_2 &= f(x_0 + c_2 h, y_0 + h a_{21} k_1) \\
k_3 &= f\big(x_0 + c_3 h, y_0 + h\,(a_{31} k_1 + a_{32} k_2)\big) \\
&\ldots \\
k_s &= f\big(x_0 + c_s h, y_0 + h\,(a_{s1} k_1 + \ldots + a_{s,s-1} k_{s-1})\big) \\
y_1 &= y_0 + h\,(b_1 k_1 + \ldots + b_s k_s)
\end{aligned}
$$

is called an s*-stage explicit Runge-Kutta method* (ERK) for (1.1).

$$c_2 = a_{21}, \quad c_3 = a_{31} + a_{32}, \quad \ldots \quad c_s = a_{s1} + \ldots + a_{s,s-1}$$

$$\|y(x_0 + h) - y_1\| \le K h^{p+1}$$

RK方法的Butche 表格

$$
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
$$

Table Low order Runge-Kutta methods

$$
\begin{array}{c|cc}
0 & & \\
1/2 & 1/2 & \\
\hline
 & 0 & 1
\end{array}
$$

Runge, order 2

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1 & 0 & 1 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 2/3 & 0 & 1/6
\end{array}
$$

Runge, order 3

$$
\begin{array}{c|ccc}
0 & & & \\
1/3 & 1/3 & & \\
2/3 & 0 & 2/3 & \\
\hline
 & 1/4 & 0 & 3/4
\end{array}
$$

Heun, order 3

4阶方法讨论

$$\sum_i b_i = b_1 + b_2 + b_3 + b_4 = 1$$
$$\sum_i b_i c_i = b_2 c_2 + b_3 c_3 + b_4 c_4 = 1/2$$
$$\sum_i b_i c_i^2 = b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = 1/3$$
$$\sum_{i,j} b_i a_{ij} c_j = b_3 a_{32} c_2 + b_4 (a_{42} c_2 + a_{43} c_3) = 1/6$$
$$\sum_i b_i c_i^3 = b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = 1/4$$
$$\sum_{i,j} b_i c_i a_{ij} c_j = b_3 c_3 a_{32} c_2 + b_4 c_4 (a_{42} c_2 + a_{43} c_3) = 1/8$$

$$\sum_{i,j} b_i a_{ij} c_j^2 = b_3 a_{32} c_2^2 + b_4 \left(a_{42} c_2^2 + a_{43} c_3^2\right) = 1/12$$
$$\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = b_4 a_{43} a_{32} c_2 = 1/24.$$

经典4阶RK、4阶Kutta方法与Gill方法

$$x(t+h)=x(t)+\frac{1}{6}(K_1+2K_2+2K_3+K_4)$$

$$\begin{cases} K_1=hf(t,x) \\ K_2=hf\left(t+\frac{1}{2}h,x+\frac{1}{2}K_1\right) \\ K_3=hf\left(t+\frac{1}{2}h,x+\frac{1}{2}K_2\right) \\ K_4=hf(t+h,x+K_3) \end{cases}$$

$$x_{n+1}=x_n+\frac{h}{6}\left[K_1+\left(2-\sqrt{2}\right)K_2+\left(2+\sqrt{2}\right)K_3+K_4\right]$$

$$\begin{cases} K_1=f\left(x_n,t_n\right) \\ K_2=f\left(t_n+\frac{1}{2}h,x_n+\frac{1}{2}hK_1\right) \\ K_3=f\left(t_n+\frac{1}{2}h,x_n+\frac{\sqrt{2}-1}{2}hK_1+\left(1-\frac{\sqrt{2}}{2}\right)hK_2\right) \\ K_4=f\left(t_n+h,x_n-\frac{\sqrt{2}}{2}hK_2+\left(1+\frac{\sqrt{2}}{2}\right)hK_3\right) \end{cases}$$

$$\begin{cases} y_{m+1}=y_m+\frac{h}{8}[k_1+3k_2+3k_3+k_4] \\ k_1=f(x_m,y_m). \\ k_2=f(x_m+\frac{1}{3}h,y_m+\frac{1}{3}k_1), \\ k_3=f(x_m+\frac{2}{3}h,y_m-\frac{1}{3}k_1+k_2), \\ k_4=f(x_m+h,y_m+k_1-k_2+k_3). \end{cases}$$

其他常见显式RK方法

六级五阶显式NYSTRÖM

0						
$\frac{1}{3}$	$\frac{1}{3}$					
$\frac{2}{5}$	$\frac{4}{25}$	$\frac{6}{25}$				
1	$\frac{1}{4}$	$-\frac{12}{4}$	$\frac{15}{4}$			
$\frac{2}{3}$	$\frac{6}{81}$	$\frac{90}{81}$	$-\frac{50}{81}$	$\frac{8}{81}$		
$\frac{4}{5}$	$\frac{6}{75}$	$\frac{36}{75}$	$\frac{10}{75}$	$\frac{8}{75}$	0	
	$\frac{23}{192}$	0	$\frac{125}{192}$	0	$-\frac{81}{192}$	$\frac{125}{192}$

六级五阶LAWSON

0						
$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$				
$\frac{1}{2}$	0	0	$\frac{1}{2}$			
$\frac{3}{4}$	0	$-\frac{3}{16}$	$\frac{6}{16}$	$\frac{9}{16}$		
1	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$-\frac{12}{7}$	$\frac{8}{7}$	
	$\frac{7}{90}$	0	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$

七级六阶BUTCHER

0							
$\frac{1}{3}$	$\frac{1}{3}$						
$\frac{2}{3}$	0	$\frac{2}{3}$					
$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$	$-\frac{1}{12}$				
$\frac{1}{2}$	$-\frac{1}{16}$	$\frac{9}{8}$	$-\frac{3}{16}$	$-\frac{3}{8}$			
$\frac{1}{2}$	0	$\frac{9}{8}$	$-\frac{3}{8}$	$-\frac{3}{4}$	$\frac{1}{2}$		
1	$\frac{9}{44}$	$-\frac{9}{11}$	$\frac{63}{44}$	$\frac{18}{11}$	0	$-\frac{16}{11}$	
	$\frac{11}{120}$	0	$\frac{27}{40}$	$\frac{27}{40}$	$-\frac{4}{15}$	$-\frac{4}{15}$	$\frac{11}{120}$

高低价嵌套变步长积分器（RKF）

轨道受力变化剧烈弧段，如大偏心率轨道

$$x(t+h)=x(t)+\frac{25}{216}K_1+\frac{1408}{2565}K_3+\frac{2197}{4104}K_4-\frac{1}{5}K_5$$

$$x(t+h)=x(t)+\frac{16}{135}K_1+\frac{6656}{12825}K_3+\frac{28561}{56430}K_4-\frac{9}{50}K_5+\frac{2}{55}K_6$$

Fehlberg 4(5)

0						
$\frac{1}{4}$	$\frac{1}{4}$					
$\frac{3}{8}$	$\frac{3}{32}$	$\frac{9}{32}$				
$\frac{12}{13}$	$\frac{1932}{2197}$	$-\frac{7200}{2197}$	$\frac{7296}{2197}$			
1	$\frac{439}{216}$	-8	$\frac{3680}{513}$	$-\frac{845}{4104}$		
$\frac{1}{2}$	$-\frac{8}{27}$	2	$-\frac{3544}{2565}$	$\frac{1859}{4104}$	$-\frac{11}{40}$	
y_1	$\frac{25}{216}$	0	$\frac{1408}{2565}$	$\frac{2197}{4104}$	$-\frac{1}{5}$	0
$\hat{y}_1$	$\frac{16}{135}$	0	$\frac{6656}{12825}$	$\frac{28561}{56430}$	$-\frac{9}{50}$	$\frac{2}{55}$

$$\begin{cases} K_1=hf(t,x) \\ K_2=hf\left(t+\frac{1}{4}h,\ x+\frac{1}{4}K_1\right) \\ K_3=hf\left(t+\frac{3}{8}h,\ x+\frac{3}{32}K_1+\frac{9}{32}K_2\right) \\ K_4=hf\left(t+\frac{12}{13}h,\ x+\frac{1932}{2197}K_1-\frac{7200}{2197}K_2+\frac{7296}{2197}K_3\right) \\ K_5=hf\left(t+h,\ x+\frac{439}{216}K_1-8K_2+\frac{3680}{513}K_3-\frac{845}{4104}K_4\right) \end{cases}$$

$$K_6=hf\left(t+\frac{1}{2}h,\ x-\frac{8}{27}K_1+2K_2-\frac{3544}{2565}K_3+\frac{1859}{4104}K_4-\frac{11}{40}K_5\right)$$

DOPRI5与RKF7(8)

Dormand-Prince 5(4) (DOPRI5)

0							
$\frac{1}{5}$	$\frac{1}{5}$						
$\frac{3}{10}$	$\frac{3}{40}$	$\frac{9}{40}$					
$\frac{4}{5}$	$\frac{44}{45}$	$-\frac{56}{15}$	$\frac{32}{9}$				
$\frac{8}{9}$	$\frac{19372}{6561}$	$-\frac{25360}{2187}$	$\frac{64448}{6561}$	$-\frac{212}{729}$			
1	$\frac{9017}{3168}$	$-\frac{355}{33}$	$\frac{46732}{5247}$	$\frac{49}{176}$	$-\frac{5103}{18656}$		
1	$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$-\frac{2187}{6784}$	$\frac{11}{84}$	
y_1	$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$-\frac{2187}{6784}$	$\frac{11}{84}$	0
$\widehat{y}_1$	$\frac{5179}{57600}$	0	$\frac{7571}{16695}$	$\frac{393}{640}$	$-\frac{92097}{339200}$	$\frac{187}{2100}$	$\frac{1}{40}$

Fehlberg 7(8)

0													
$\frac{2}{27}$	$\frac{2}{27}$												
$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{12}$											
$\frac{1}{6}$	$\frac{1}{24}$	0	$\frac{1}{8}$										
$\frac{5}{12}$	$\frac{5}{12}$	0	$-\frac{25}{16}$	$\frac{25}{16}$									
$\frac{1}{2}$	$\frac{1}{20}$	0	0	$\frac{1}{4}$	$\frac{1}{5}$								
$\frac{5}{6}$	$-\frac{25}{108}$	0	0	$\frac{125}{108}$	$-\frac{65}{27}$	$\frac{125}{54}$							
$\frac{1}{6}$	$\frac{31}{300}$	0	0	0	$\frac{61}{225}$	$-\frac{2}{9}$	$\frac{13}{900}$						
$\frac{2}{3}$	2	0	0	$-\frac{53}{6}$	$\frac{704}{45}$	$-\frac{107}{9}$	$\frac{67}{90}$	3					
$\frac{1}{3}$	$-\frac{91}{108}$	0	0	$\frac{23}{108}$	$-\frac{976}{135}$	$\frac{311}{54}$	$-\frac{19}{60}$	$\frac{17}{6}$	$-\frac{1}{12}$				
1	$\frac{2383}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{301}{82}$	$\frac{2133}{4100}$	$\frac{45}{82}$	$\frac{45}{164}$	$\frac{18}{41}$			
0	$\frac{3}{205}$	0	0	0	0	$-\frac{6}{41}$	$-\frac{3}{205}$	$-\frac{3}{41}$	$\frac{3}{41}$	$\frac{6}{41}$	0		
1	$-\frac{1777}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{289}{82}$	$\frac{2193}{4100}$	$\frac{51}{82}$	$\frac{33}{164}$	$\frac{19}{41}$	0	1	
y_1	$\frac{41}{840}$	0	0	0	0	$\frac{34}{105}$	$\frac{9}{35}$	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{9}{280}$	$\frac{41}{840}$	0	0
$\widehat{y}_1$	0	0	0	0	0	$\frac{34}{105}$	$\frac{9}{35}$	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{9}{280}$	0	$\frac{41}{840}$	$\frac{41}{840}$

隐式RK方法及其存在性

Definition Let $b_i,\ a_{ij}\ (i,j=1,\ldots,s)$ be real numbers and let c_i be defined by (1.9). The method

$$k_i = f\Big(x_0 + c_i h, y_0 + h\sum_{j=1}^{s} a_{ij}k_j\Big) \qquad i=1,\ldots,s$$

$$y_1 = y_0 + h\sum_{i=1}^{s} b_i k_i$$

is called an *s-stage Runge-Kutta method.* When $a_{ij}=0$ for $i\le j$ we have an explicit (ERK) method. If $a_{ij}=0$ for $i<j$ and at least one $a_{ii}\neq 0$, we have a *diagonal implicit Runge-Kutta method* (DIRK). If in addition all diagonal elements are identical ($a_{ii}=\gamma$ for $i=1,\ldots,s$), we speak of a *singly diagonal* implicit (SDIRK) method. In all other cases we speak of an *implicit* Runge-Kutta method (IRK).

Theorem *Let $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be continuous and satisfy a Lipschitz condition with constant L (with respect to y). If*

$$h < \frac{1}{L\ \max_i \sum_j |a_{ij}|}$$

there exists a unique solution of , which can be obtained by iteration. If $f(x,y)$ is p times continuously differentiable, the functions k_i (as functions of h) are also in C^p .

范例

$$y_1 = y_0 + hf(x_1, y_1)$$

隐式欧拉折线

$$k_1 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} k_1\right)$$
$$y_1 = y_0 + hk_1.$$

隐式欧拉中点格式

$$k_1 = f(x_0, y_0)$$
$$k_2 = f\left(x_0 + \frac{2}{3} h, y_0 + \frac{h}{3} (k_1 + k_2)\right)$$
$$y_1 = y_0 + \frac{h}{4} (k_1 + 3k_2).$$

Hammer & Hollingsworth

典型隐式RK方法

Table Kuntzmann & Butcher method, order 6

$\frac{1}{2}-\frac{\sqrt{15}}{10}$	$\frac{5}{36}$	$\frac{2}{9}-\frac{\sqrt{15}}{15}$	$\frac{5}{36}-\frac{\sqrt{15}}{30}$
$\frac{1}{2}$	$\frac{5}{36}+\frac{\sqrt{15}}{24}$	$\frac{2}{9}$	$\frac{5}{36}-\frac{\sqrt{15}}{24}$
$\frac{1}{2}+\frac{\sqrt{15}}{10}$	$\frac{5}{36}+\frac{\sqrt{15}}{30}$	$\frac{2}{9}+\frac{\sqrt{15}}{15}$	$\frac{5}{36}$
	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{5}{18}$

Table Kuntzmann & Butcher method, order 8

$\frac{1}{2}-\omega_2$	ω_1	$\omega_1'-\omega_3+\omega_4'$	$\omega_1'-\omega_3-\omega_4'$	$\omega_1-\omega_5$
$\frac{1}{2}-\omega_2'$	$\omega_1-\omega_3'+\omega_4$	ω_1'	$\omega_1'-\omega_5'$	$\omega_1-\omega_3'-\omega_4$
$\frac{1}{2}+\omega_2'$	$\omega_1+\omega_3'+\omega_4$	$\omega_1'+\omega_5'$	ω_1'	$\omega_1+\omega_3'-\omega_4$
$\frac{1}{2}+\omega_2$	$\omega_1+\omega_5$	$\omega_1'+\omega_3+\omega_4'$	$\omega_1'+\omega_3-\omega_4'$	ω_1
	$2\omega_1$	$2\omega_1'$	$2\omega_1'$	$2\omega_1$

$$\omega_1=\frac{1}{8}-\frac{\sqrt{30}}{144},\qquad \omega_1'=\frac{1}{8}+\frac{\sqrt{30}}{144},$$

$$\omega_2=\frac{1}{2}\sqrt{\frac{15+2\sqrt{30}}{35}},\qquad \omega_2'=\frac{1}{2}\sqrt{\frac{15-2\sqrt{30}}{35}},$$

$$\omega_3=\omega_2\left(\frac{1}{6}+\frac{\sqrt{30}}{24}\right),\qquad \omega_3'=\omega_2'\left(\frac{1}{6}-\frac{\sqrt{30}}{24}\right),$$

$$\omega_4=\omega_2\left(\frac{1}{21}+\frac{5\sqrt{30}}{168}\right),\qquad \omega_4'=\omega_2'\left(\frac{1}{21}-\frac{5\sqrt{30}}{168}\right),$$

$$\omega_5=\omega_2-2\omega_3,\qquad \omega_5'=\omega_2'-2\omega_3'.$$

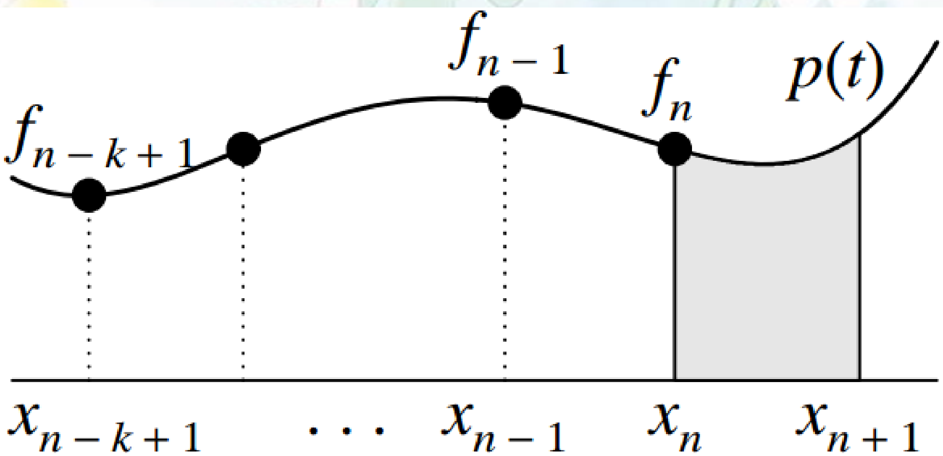
主要内容

- 单步法
- 多步法
- 辛积分器
- 精细积分法

显式Adams-Bashforth多步法

$$y' = f(x, y), \quad y(x_0) = y_0$$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f\big(t, y(t)\big)\, dt.$$



Explicit Adams methods

显式Adams多步法

$$\nabla^0 f_n = f_n, \qquad \nabla^{j+1} f_n = \nabla^j f_n - \nabla^j f_{n-1}$$

$$p(t) = p(x_n + sh) = \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j f_n$$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} p(t)\, dt$$

$$y_{n+1} = y_n + h \sum_{j=0}^{k-1} \gamma_j \nabla^j f_n \qquad \gamma_j = (-1)^j \int_0^1 \binom{-s}{j} ds$$

低阶Adams方法

Coefficients for the explicit Adams methods

j	0	1	2	3	4	5	6	7	8
γ_j	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$	$\frac{19087}{60480}$	$\frac{5257}{17280}$	$\frac{1070017}{3628800}$

$$k=1: \qquad y_{n+1} = y_n + hf_n \qquad \text{(explicit Euler method)}$$

$$k=2: \qquad y_{n+1} = y_n + h\Big(\frac{3}{2}f_n - \frac{1}{2}f_{n-1}\Big)$$

$$k=3: \qquad y_{n+1} = y_n + h\Big(\frac{23}{12}f_n - \frac{16}{12}f_{n-1} + \frac{5}{12}f_{n-2}\Big)$$

$$k=4: \qquad y_{n+1} = y_n + h\Big(\frac{55}{24}f_n - \frac{59}{24}f_{n-1} + \frac{37}{24}f_{n-2} - \frac{9}{24}f_{n-3}\Big).$$

系数递推关系

$$G(t)=\sum_{j=0}^{\infty}(-t)^j\int_0^1\binom{-s}{j}\,ds=\int_0^1\sum_{j=0}^{\infty}(-t)^j\binom{-s}{j}\,ds$$
$$=\int_0^1(1-t)^{-s}\,ds=-\frac{t}{(1-t)\log(1-t)}.$$

$$-\frac{\log(1-t)}{t}G(t)=\frac{1}{1-t}$$

$$(1+\frac{1}{2}t+\frac{1}{3}t^2+\ldots)(\gamma_0+\gamma_1 t+\gamma_2 t^2+\ldots)=(1+t+t^2+\ldots)$$

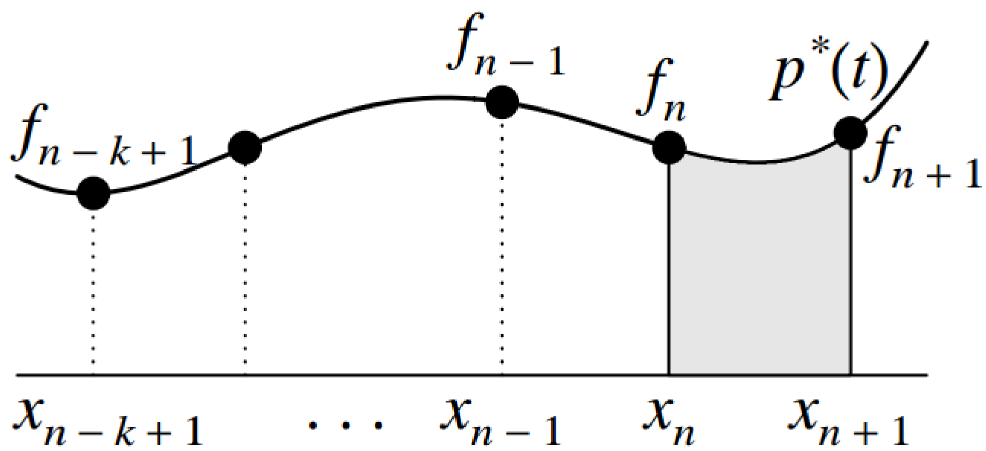
$$\gamma_m+\frac{1}{2}\gamma_{m-1}+\frac{1}{3}\gamma_{m-2}+\ldots+\frac{1}{m+1}\gamma_0=1$$

隐式Adams-Moulton多步法

$$p^*(t) = p^*(x_n + sh) = \sum_{j=0}^{k} (-1)^j \binom{-s+1}{j} \nabla^j f_{n+1}$$

$$y_{n+1} = y_n + h \sum_{j=0}^{k} \gamma_j^* \nabla^j f_{n+1}$$

$$\gamma_j^* = (-1)^j \int_0^1 \binom{-s+1}{j} ds$$



Implicit Adams methods

低阶隐式Adams方法

Coefficients for the implicit Adams methods

j	0	1	2	3	4	5	6	7	8
γ_j^*	1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{3}{160}$	$-\frac{863}{60480}$	$-\frac{275}{24192}$	$-\frac{33953}{3628800}$

$$y_{n+1} = y_n + h\big(\beta_k f_{n+1} + \ldots + \beta_0 f_{n-k+1}\big)$$

$$k = 0: \qquad y_{n+1} = y_n + hf_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

$$k = 1: \qquad y_{n+1} = y_n + h\Big(\frac{1}{2}f_{n+1} + \frac{1}{2}f_n\Big)$$

$$k = 2: \qquad y_{n+1} = y_n + h\Big(\frac{5}{12}f_{n+1} + \frac{8}{12}f_n - \frac{1}{12}f_{n-1}\Big)$$

$$k = 3: \qquad y_{n+1} = y_n + h\Big(\frac{9}{24}f_{n+1} + \frac{19}{24}f_n - \frac{5}{24}f_{n-1} + \frac{1}{24}f_{n-2}\Big)$$

联合显式隐式预测校正方法PECE

P: compute the predictor $\widehat{y}_{n+1} = y_n + h\sum_{j=0}^{k-1}\gamma_j \nabla^j f_n$ by the explicit Adams method (1.5); this already yields a reasonable approximation to $y(x_{n+1})$;

E: evaluate the function at this approximation: $\widehat{f}_{n+1} = f(x_{n+1}, \widehat{y}_{n+1})$;

C: apply the corrector formula

$$y_{n+1} = y_n + h\big(\beta_k \widehat{f}_{n+1} + \beta_{k-1} f_n + \ldots + \beta_0 f_{n-k+1}\big)$$

to obtain y_{n+1}.

E: evaluate the function anew, i.e., compute $f_{n+1} = f(x_{n+1}, y_{n+1})$.

二阶微分方程的多步法问题

$$y'' = f(x, y, y')$$

We rewrite in the usual way as a first order system and apply a multistep method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i}$$

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i f(x_{n+i}, y_{n+i}, y'_{n+i})$$

如果右函数不含有一次导数项

$$\sum_{i=0}^{2k} \widehat{\alpha}_i y_{n+i} = h^2 \sum_{i=0}^{2k} \widehat{\beta}_i f(x_{n+i}, y_{n+i})$$

Stömer显式方法

$$y(x+h) = y(x) + hy'(x) + h^2 \int_0^1 (1-s) f\big(x+sh, y(x+sh)\big)\, ds.$$

$$\begin{aligned} &y(x+h) - 2y(x) + y(x-h) \\ &\qquad = h^2 \int_0^1 (1-s)\Big(f\big(x+sh, y(x+sh)\big) + f\big(x-sh, y(x-sh)\big)\Big)\, ds. \end{aligned}$$

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \sum_{j=0}^{k-1} \sigma_j \nabla^j f_n$$

$$\sigma_j = (-1)^j \int_0^1 (1-s)\left(\binom{-s}{j} + \binom{s}{j}\right) ds$$

系数表及低阶方法

Coefficients of the method (10.10)

j	0	1	2	3	4	5	6	7	8	9
σ_j	1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{19}{240}$	$\frac{3}{40}$	$\frac{863}{12096}$	$\frac{275}{4032}$	$\frac{33953}{518400}$	$\frac{8183}{129600}$

$$k=2: \quad y_{n+1}-2y_n+y_{n-1}=h^2 f_n$$

$$k=3: \quad y_{n+1}-2y_n+y_{n-1}=h^2\left(\frac{13}{12}f_n-\frac{1}{6}f_{n-1}+\frac{1}{12}f_{n-2}\right)$$

$$k=4: \quad y_{n+1}-2y_n+y_{n-1}=h^2\left(\frac{7}{6}f_n-\frac{5}{12}f_{n-1}+\frac{1}{3}f_{n-2}-\frac{1}{12}f_{n-3}\right)$$

Cowell隐式方法

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \sum_{j=0}^{k} \sigma_j^* \nabla^j f_{n+1}$$

$$\sigma_j^* = (-1)^j \int_0^1 (1-s)\left(\binom{-s+1}{j} + \binom{s+1}{j}\right) ds$$

Coefficients of the implicit method

j	0	1	2	3	4	5	6	7	8	9
σ_j^*	1	−1	$\frac{1}{12}$	0	$\frac{-1}{240}$	$\frac{-1}{240}$	$\frac{-221}{60480}$	$\frac{-19}{6048}$	$\frac{-9829}{3628800}$	$\frac{-407}{172800}$

卫星轨道计算常用数值方法

- Adams显式隐式校正方法（PANDA,GAMIT）
  RK起步
- Adams-Cowell方法（EPOS）
- Utopia,SODP（KSG方法，属于Adams-cowell类型）
  迭代自起步。
- 配置法（Bernese）

在人造卫星轨道计算中，可以混合使用Adams方法与Cowell方法，比单纯的Adams方法更有效。

主要内容

- 单步法
- 多步法
- 辛积分器
- 精细积分法

相流保体积（以下内容为选学）

$$\frac{dy}{dt} = f(y)$$

$$\varphi_t(y^0) := y(t, 0, y^0).$$

$$\mathrm{Vol}\,(\varphi_t(A)) = \int_{\varphi_t(A)} dy = \int_A \left|\det\left(\frac{\partial y}{\partial y^0}(t, 0, y^0)\right)\right| dy^0$$

Theorem *Consider the system* *with continuously differentiable function* $f(y)$.

a) *For a set* $A \subset \mathbb{R}^n$ *the total volume of* $\varphi_t(A)$ *satisfies*

$$\mathrm{Vol}\,(\varphi_t(A)) = \int_A \exp\left(\int_0^t \mathrm{tr}\,(f'(y(s, 0, y^0)))\, ds\right) dy^0.$$

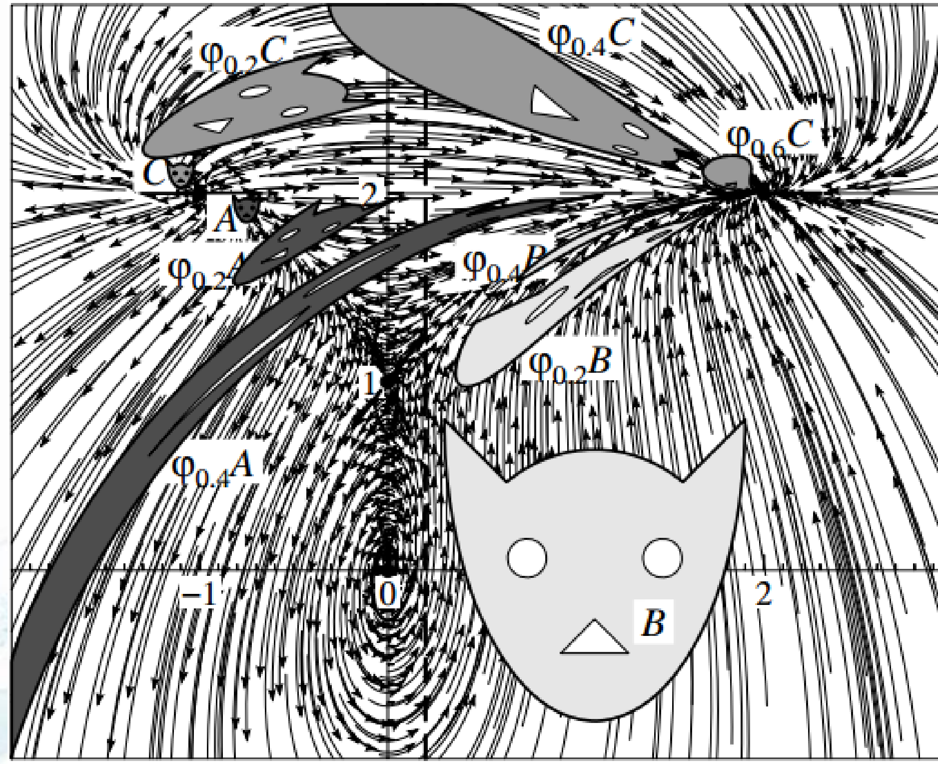
b) *If* $\mathrm{tr}\,(f'(y)) = 0$ *along the solution, the flow is volume-preserving, i.e.,* $\mathrm{Vol}\,(\varphi_t(A)) = \mathrm{Vol}\,(A)$. □

非保体积相流范例

$$y_1' = \frac{1}{3}(y_1 - y_2)(1 - y_1 - y_2)$$
$$y_2' = y_1(2 - y_2)$$

$$f'(y) = \begin{pmatrix} (1-2y_1)/3 & (2y_2-1)/3 \\ 2-y_2 & -y_1 \end{pmatrix}$$

$$\mathrm{tr}\,(f'(y)) = (1-5y_1)/3$$



Transformation of three sets under a flow

The trace of $f'(y)$ changes sign at the line $y_1 = 1/5$. To its left the volume increases, to the right we have decreasing volumes. This can clearly be seen in

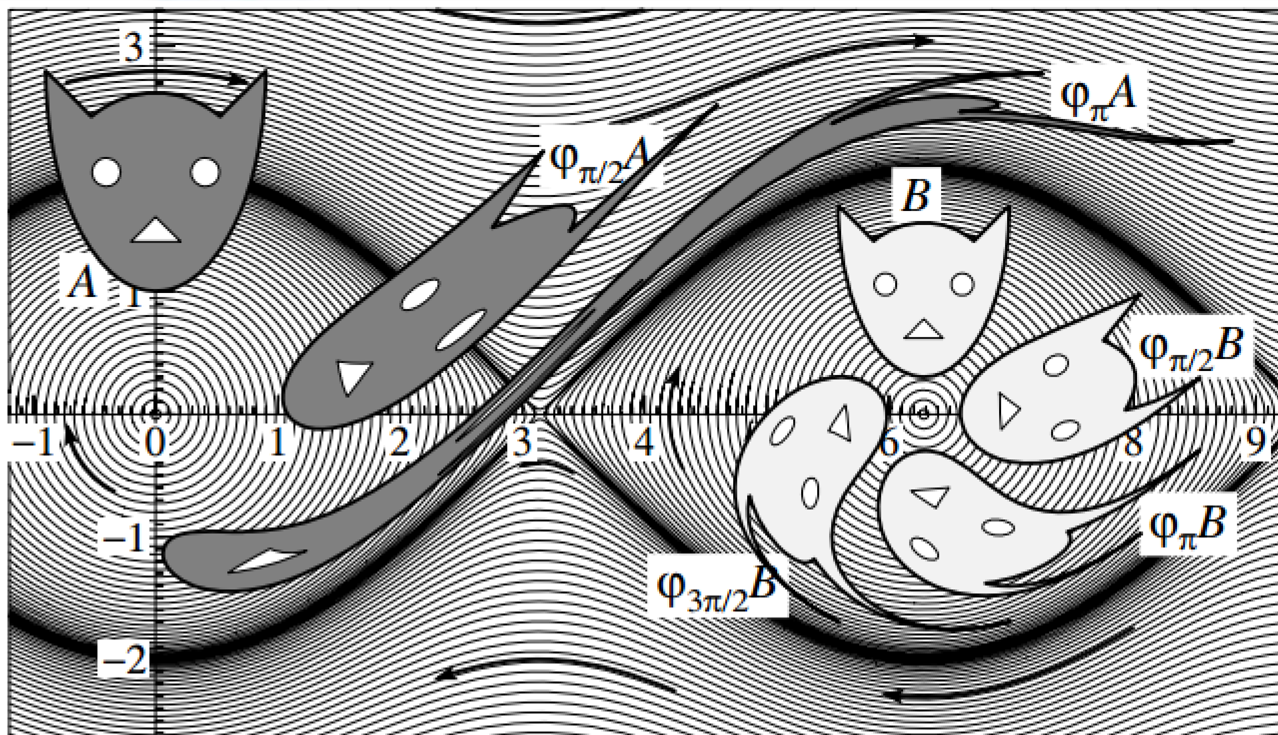
Example Fig. 14.2 shows, for the two-dimensional system , the transformations which three sets A, B, C [2] undergo when t passes from 0 to $0.2, 0.4$ and (for C) 0.6. It can be observed that these sets quickly lose very much of their beauty.

保体积相流范例

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -\sin y_1$$

$$f'(y) = \begin{pmatrix} 0 & 1 \\ -\cos y_1 & 0 \end{pmatrix}$$

$$\operatorname{tr}\left(f'(y)\right) = 0$$



辛结构与哈密顿积分器

$$\begin{cases} \dfrac{d\boldsymbol{q}}{dt} = \dfrac{\partial H}{\partial \boldsymbol{p}} \\ \dfrac{d\boldsymbol{p}}{dt} = -\dfrac{\partial H}{\partial \boldsymbol{q}} \end{cases}$$

$$\boldsymbol{J} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{v} = \boldsymbol{J}\left(\frac{\partial H}{\partial \boldsymbol{v}}\right)$$

$$\boldsymbol{v} = \left(\boldsymbol{q}^T, \boldsymbol{p}^T\right)^T$$

在状态向量的形式下，定常正则变换可以用 $2n$ 维向量的变换描述：

$$\varsigma = \varsigma(\boldsymbol{v}), \varsigma = \varsigma\left(\boldsymbol{Q}^T, \boldsymbol{P}^T\right)^T$$

其逆变换也是定常的，把逆变换和变换分别带入正则方程得到

$$\dot{\varsigma} = \left(\frac{\partial \varsigma}{\partial \mathbf{v}}\right)^T \boldsymbol{J}\left(\frac{\partial \varsigma}{\partial \boldsymbol{v}}\right)\frac{\partial K}{\partial \boldsymbol{v}}$$

$$\boldsymbol{S} = \left(\frac{\partial \varsigma}{\partial \boldsymbol{v}}\right)^T$$

$$\boldsymbol{S}^T \boldsymbol{J} \boldsymbol{S} = \boldsymbol{J}$$

Kang Feng giving a talk at an international conference

辛映射性质

封闭性。辛映射的复合依然是辛映射。即 a 与 b 是任意的辛映射，则有唯一确定的辛映射 $a \circ b = c$ 也是辛映射。

满足结合律。即对于辛映射 a, b, c，满足 $(a \circ b) \circ c = a \circ (b \circ c)$。

有单位元素存在。存在单位映射 1，有 $1 \circ a = a = a \circ 1$。实际上即不进行正则变换，或者说变换就是其本身。

辛映射存在逆映射。即对于映射 f，总存在逆映射 a^{-1}，使得 $a^{-1} \circ a = 1 = a \circ a^{-1}$。

因此辛映射（或辛矩阵）构成一个群，且有子群，如行列式为 1 的辛矩阵构成子群。

偶次微分流形的辛结构（或叫辛构造）是一个闭的非退化的微分 2 形式 ω^2，在流形 $M^{2n} = \left\{ \left(\boldsymbol{p}^T, \boldsymbol{q}^T \right)^T \right\}$ 中，ω^2 可以表示为

$$\omega^2 = d\boldsymbol{p} \wedge d\boldsymbol{q}$$

在辛流形 (M^{2n}, ω) 上，哈密顿函数 H 的矢量场 $\boldsymbol{J}dH$ 给出一个单参数微分同胚群 g^t : $M^{2n} \to M^{2n}$：

$$\frac{d}{dt}|_{t=0} g^t \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} = \boldsymbol{J}dH \left(\begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} \right)$$

群 g^t 称为具有哈密顿函数 H 的哈密顿相流。哈密顿相流具保持辛结构特性，即

$$\left(g^t\right)^* \omega^2 = \omega^2$$

当 $n = 1$ 时候 $M^{2n} = R^2$，就是大家熟悉的相流保面积，即刘维尔定理。

哈密顿系统的辛RK积分器

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}, \quad \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad i = 1, 2, \cdots, n.$$

$$z = \begin{bmatrix} p_1 \\ \vdots \\ p_n \\ q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \\ \vdots \\ z_{2n} \end{bmatrix}, \quad H_z = \begin{bmatrix} H_{z_1} \\ \vdots \\ H_{z_n} \\ \vdots \\ H_{z_{2n}} \end{bmatrix}$$

$$\psi_h : \begin{array}{ccc} \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2n} \\ (p_0, q_0) & \longmapsto & (p_1, q_1) \end{array}$$

$$J = J_{2n} = \begin{bmatrix} O & I_n \\ -I_n & O \end{bmatrix}, \quad J' = J^{-1} = -J,$$

单步的差分格式看成时间t(k)到t(k+1)的变换，称为推进映射，如果推进映射是辛的，则称此格式为辛格式

哈密顿系统的辛RK积分器

Theorem *If the $s \times s$ matrix M with elements*

$$m_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j, \qquad i, j = 1, \dots, s$$

satisfies $M = 0$, then the Runge-Kutta method is symplectic.

s-stage R–K method for has the following form:

$$z^{k+1} = z^k + h \sum_{i=1}^{s} b_i f(Y_i),$$

$$Y_i = z^k + h \sum_{j=1}^{s} a_{ij} f(Y_j), \quad 1 \le i \le s$$

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ c_2 & a_{21} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

辛RK积分器范例

$$z_{m+1} = z_m + hf(k_1), \quad k_1 = z_m + \frac{h}{2} f(k_1)$$ 中点 Euler 公式

$$\begin{aligned}
z_{m+1} &= z_m + \frac{h}{2}\left(f(k_1) + f(k_2)\right), \\
k_1 &= z_m + h\Big(\frac{1}{4} f(k_1) + \Big(\frac{1}{4} - \frac{1}{6}\sqrt{3}\Big) f(k_2)\Big), \\
k_2 &= z_m + h\left(\Big(\frac{1}{4} + \frac{1}{6}\sqrt{3}\Big) f(k_1) + \frac{1}{4} f(k_2)\right)
\end{aligned}$$

2 级 4 阶隐式 Runge–Kutta 格式

$$\begin{aligned}
z_{m+1} &= z_m + \frac{h}{2}\left(f(k_1) + f(k_2)\right) \\
k_1 &= z_m + \frac{h}{4} f(k_1), \\
k_2 &= z_m + \frac{h}{2} f(k_1) + \frac{h}{4} f(k_2)
\end{aligned}$$

2 级 2 阶对角隐式 Runge–Kutta 格式

可分哈密顿系统

如果哈密顿函数可以表示为：

$$\boldsymbol{H}\left(\boldsymbol{z}\right)=\boldsymbol{H}\left(\boldsymbol{q},p\right)=\boldsymbol{U}\left(\boldsymbol{p}\right)+\boldsymbol{V}\left(\boldsymbol{p}\right)$$

则称哈密顿系统是可分的，可分哈密顿系统的正则方程为：

$$\begin{cases}\dfrac{d\boldsymbol{q}}{dt}=\dfrac{\partial \boldsymbol{U}\left(\boldsymbol{p}\right)}{\partial \boldsymbol{p}}\\ \dfrac{d\boldsymbol{p}}{dt}=-\dfrac{\partial \boldsymbol{V}\left(\boldsymbol{q}\right)}{\partial \boldsymbol{q}}\end{cases}$$

这里给出一种由日本学者 Yoshida 构造的四阶显式辛格式：

$$\boldsymbol{x}^1=\boldsymbol{p}^n+c_1h\left(-\frac{\partial \boldsymbol{V}\left(\boldsymbol{q}^n\right)}{\partial \boldsymbol{q}}\right),\boldsymbol{y}^1=\boldsymbol{q}^n+d_1h\frac{\partial \boldsymbol{U}\left(\boldsymbol{x}^1\right)}{\partial \boldsymbol{p}}$$

$$\boldsymbol{x}^2=\boldsymbol{x}^1+c_2h\left(-\frac{\partial \boldsymbol{V}\left(\boldsymbol{y}^1\right)}{\partial \boldsymbol{q}}\right),\boldsymbol{y}^2=\boldsymbol{y}^1+d_2h\frac{\partial \boldsymbol{U}\left(\boldsymbol{x}^2\right)}{\partial \boldsymbol{p}}$$

$$\boldsymbol{x}^3=\boldsymbol{x}^2+c_3h\left(-\frac{\partial \boldsymbol{V}\left(\boldsymbol{y}^2\right)}{\partial \boldsymbol{q}}\right),\boldsymbol{y}^3=\boldsymbol{y}^2+d_2h\frac{\partial \boldsymbol{U}\left(\boldsymbol{x}^3\right)}{\partial \boldsymbol{p}}$$

$$\boldsymbol{p}^{n+1}=\boldsymbol{x}^3+c_4h\left(-\frac{\partial \boldsymbol{V}\left(\boldsymbol{y}^3\right)}{\partial \boldsymbol{q}}\right),\boldsymbol{q}^{n+1}=\boldsymbol{y}^3+d_4h\frac{\partial \boldsymbol{U}\left(\boldsymbol{p}^{n+1}\right)}{\partial \boldsymbol{p}}$$

$$\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}0\\\alpha\\\beta\\\alpha\end{pmatrix},\qquad \begin{pmatrix}d_1\\d_2\\d_3\\d_4\end{pmatrix}=\begin{pmatrix}\frac{\alpha}{2}\\\frac{\alpha+\beta}{2}\\\frac{\alpha+\beta}{2}\\\frac{\alpha}{2}\end{pmatrix}$$

$$\alpha=\left(2-2^{\frac{1}{3}}\right)^{-1},\beta=1-2\alpha$$

谐振子四种简单积分比较

$$H(p,q) = \frac{1}{2}\left(p^2 + k^2 q^2\right)$$

$$\dot{p} = -k^2 q, \qquad \dot{q} = p$$

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

The explicit Euler method

$$\begin{pmatrix} p_m \\ q_m \end{pmatrix} = \begin{pmatrix} 1 & -hk^2 \\ h & 1 \end{pmatrix} \begin{pmatrix} p_{m-1} \\ q_{m-1} \end{pmatrix}, \quad h = \frac{\pi}{8k}, \ m = 1, \ldots, 16;$$

the implicit (or backward) Euler method

$$\begin{pmatrix} p_m \\ q_m \end{pmatrix} = \frac{1}{1+h^2k^2} \begin{pmatrix} 1 & -hk^2 \\ h & 1 \end{pmatrix} \begin{pmatrix} p_{m-1} \\ q_{m-1} \end{pmatrix}, \quad h = \frac{\pi}{8k}, \ m = 1, \ldots, 16;$$

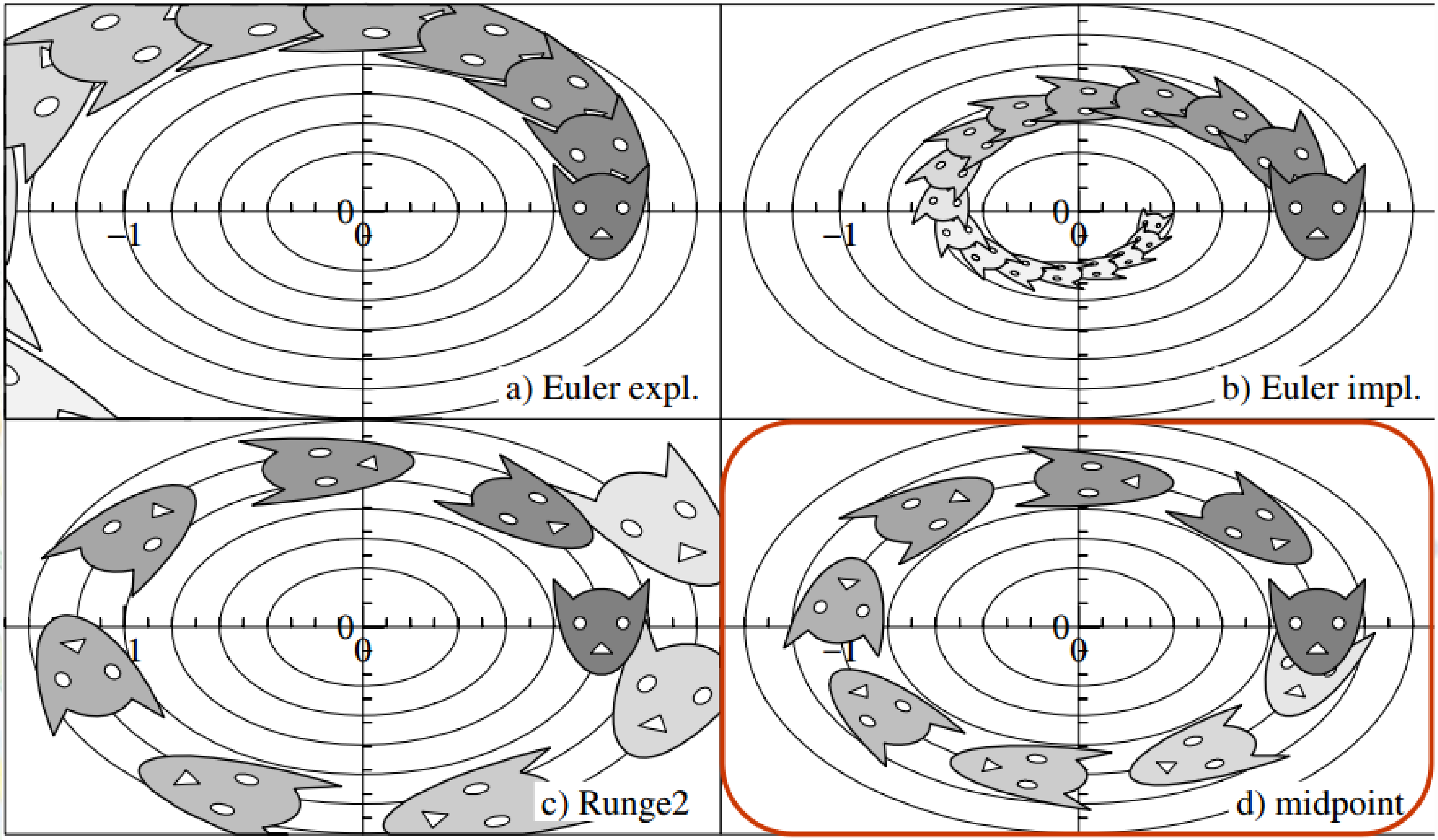
Runge's method (1.4) of order 2

$$\begin{pmatrix} p_m \\ q_m \end{pmatrix} = \begin{pmatrix} 1 - \frac{h^2k^2}{2} & -hk^2 \\ h & 1 - \frac{h^2k^2}{2} \end{pmatrix} \begin{pmatrix} p_{m-1} \\ q_{m-1} \end{pmatrix}, \quad h = \frac{\pi}{4k}, \ m = 1, \ldots, 8;$$

the implicit midpoint rule (7.4) of order 2

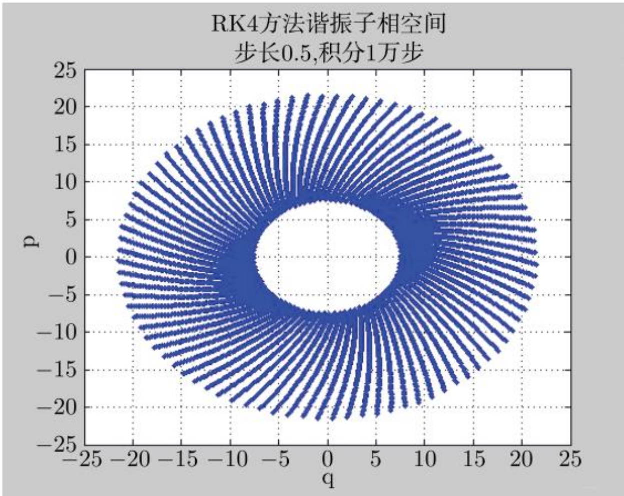
$$\begin{pmatrix} p_m \\ q_m \end{pmatrix} = \frac{1}{1+\frac{h^2k^2}{4}} \begin{pmatrix} 1 - \frac{h^2k^2}{4} & -hk^2 \\ h & 1 - \frac{h^2k^2}{4} \end{pmatrix} \begin{pmatrix} p_{m-1} \\ q_{m-1} \end{pmatrix}, \quad h = \frac{\pi}{4k}, \ m = 1, \ldots, 8.$$

谐振子积分比较



Destruction of symplecticity of a Hamiltonian flow, $k = (\sqrt{5}+1)/2$

谐振子哈密顿积分(Song,2012)

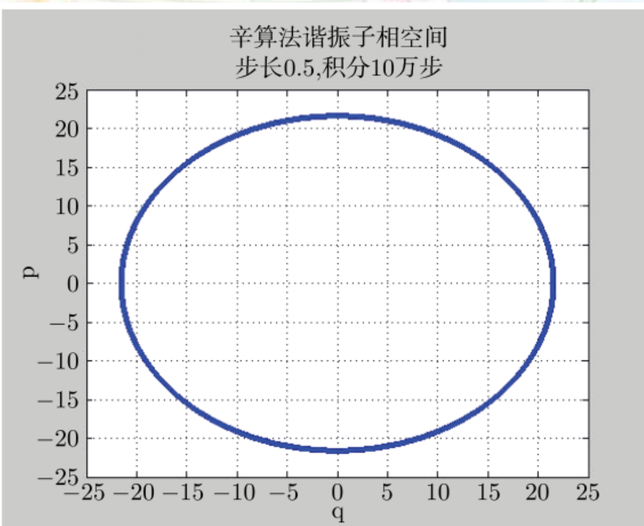


).1 普通 RK 方法计算谐振子相图（步长 0.5，积分 1 万步）

$$L = T - V = \frac{1}{2}m\dot{x} - \frac{1}{2}kx^2$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

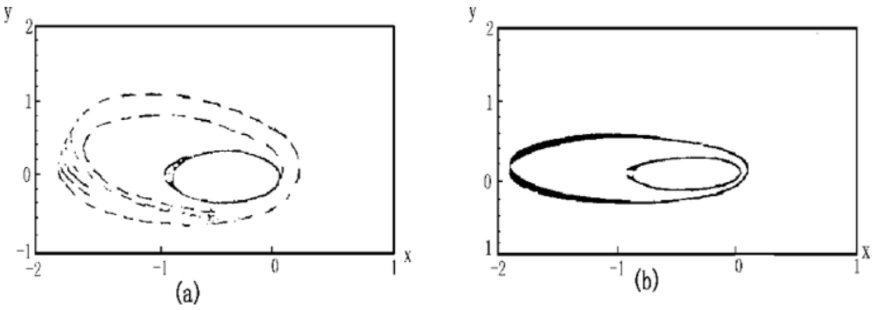
$$H = p\dot{x} - L = \frac{1}{m}p^2 + \frac{1}{2}kx^2$$



辛算法（哈密顿）计算的谐振子相图（步长 0.5，积分 10 万步）

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{p}{m} \\ \dfrac{dp}{dt} = -kx \end{cases}$$

辛算法在的轨道计算中定量分析（Liu）



Comparison of calculation of Keplerian motion by R–K and symplectic methods.

开普勒轨道

Errors of trajectories with nonspherical perturbation of the Earth$\Delta(M+\omega)$

method	N of steps / circle	100 circles	1000 circles	10000 circles
FKF7(8)	100	1.5 E − 10	1.4 E − 08	1.3 E − 06
SY6	50	0.5 E − 09	0.6 E − 08	1.0 E − 07
RKH	100	0.9 E − 11	0.9 E − 10	0.9 E − 09

LAGEOS 卫星

Errors of trajectories with perturbation of atmospheric resistance$\Delta(M+\omega)$

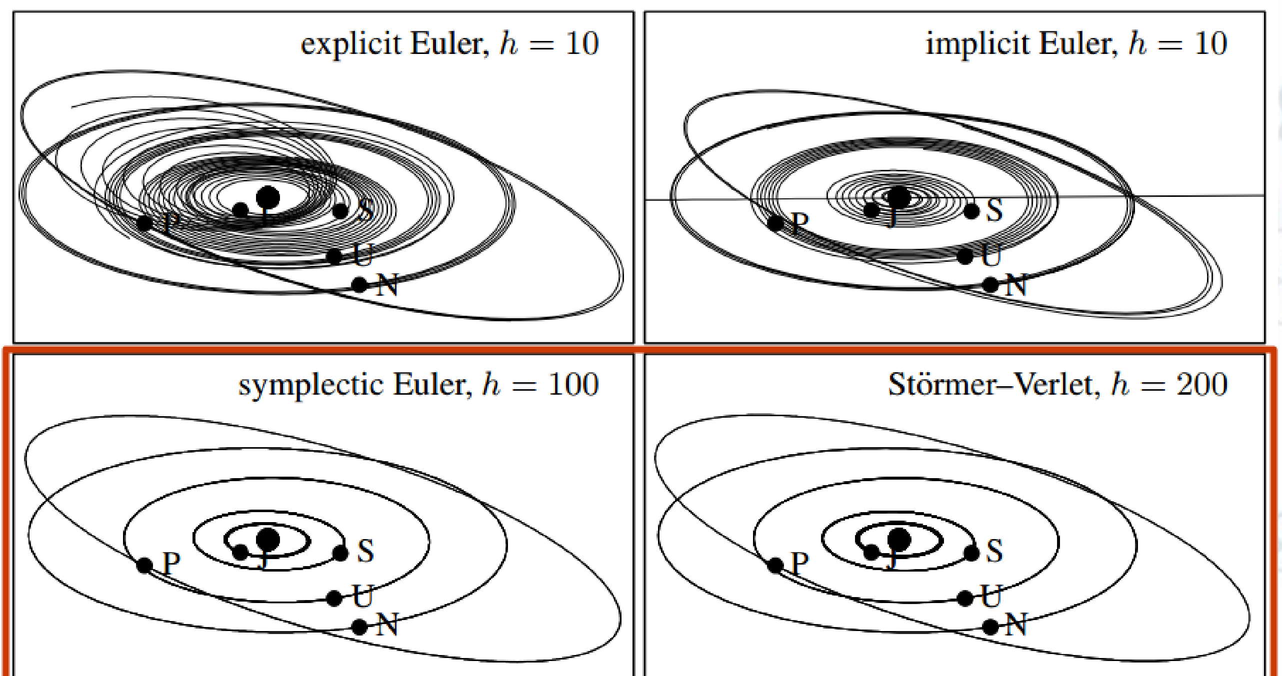
method	N of steps / circle	100 circles	1000 circles	10000 circles
FKF7(8)	100	1.4 E − 410	1.3 E − 08	1.3 E − 06
SY6	50	0.6 E − 09	0.7 E − 08	1.0 E − 07
RKH	100	2.1 E − 11	3.5 E − 10	6.2 E − 09

E. Hairer

The outer solar system

$$H(p,q) = \frac{1}{2}\sum_{i=0}^{5}\frac{1}{m_i}p_i^T p_i - G\sum_{i=1}^{5}\sum_{j=0}^{i-1}\frac{m_i m_j}{\|q_i - q_j\|}$$

planet	mass	initial position	initial velocity
Jupiter	$m_1 = 0.000954786104043$	−3.5023653 −3.8169847 −1.5507963	0.00565429 −0.00412490 −0.00190589
Saturn	$m_2 = 0.000285583733151$	9.0755314 −3.0458353 −1.6483708	0.00168318 0.00483525 0.00192462
Uranus	$m_3 = 0.0000437273164546$	8.3101420 −16.2901086 −7.2521278	0.00354178 0.00137102 0.00055029
Neptune	$m_4 = 0.0000517759138449$	11.4707666 −25.7294829 −10.8169456	0.00288930 0.00114527 0.00039677
Pluto	$m_5 = 1/(1.3 \cdot 10^8)$	−15.5387357 −25.2225594 −3.1902382	0.00276725 −0.00170702 −0.00136504



积分时间200000days

主要内容

- 单步法
- 多步法
- 辛积分器
- 精细积分法

精细积分方法

钟万勰院士提出的齐次线性定常微分系统的精细积分方法。精细积分法看上去简单，但是对于这一类微分方程可以获得非常高精度的计算结果，亦既“计算机上的精确解”。

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}_0$$

对于不满足精细积分法的方程，常常可以在此基础上以摄动法等手段进行处理 。精细积分法在控制论与弹性力学中已经取得了很重要的应用。

精细积分法

$$\exp(\mathbf{A}t) = \mathbf{I}_n + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots$$

$$\mathbf{x}_1 = \mathbf{T}\mathbf{x}_0 = \exp(\mathbf{A}t)\,\mathbf{x}_0$$

$$\mathbf{x}_1 = \mathbf{T}\mathbf{x}_0, \mathbf{x}_2 = \mathbf{T}\mathbf{x}_2, \cdots, \mathbf{x}_{k+1} = \mathbf{T}\mathbf{x}_k, \cdots$$

问题便转到计算矩阵T上来，要求其精确性较高。矩阵指数计算有两个要点：（1）利用指数函数的加法定理，（2）将注意力集中在增量上，而不是全量

$$\exp(\mathbf{A}\eta) = \left(\exp\left(\mathbf{A}\frac{\eta}{m}\right)\right)^m$$

精细积分法

其中m为任意整数，这里可以选m ＝ 2N。假设N=20，则m ＝ 1048576。本来每段区间就不大，则τ ＝ η/m是一个非常小的区段。在该很小的区段上，幂级数前几项展开已经足够，此时矩阵指数与单位矩阵非常接近，可以写为

$$\exp\left(\mathbf{A}\tau\right) \approx \mathbf{I}_n + \mathbf{T}_a$$

$$\mathbf{T}_a = \mathbf{A}\tau + \frac{\left(\mathbf{A}\tau\right)^2}{2}\left[\mathbf{I}_n + \frac{\mathbf{A}\tau}{3} + \frac{\left(\mathbf{A}\tau\right)^2}{12}\right]$$

精细积分法

在计算机中直观重要的一点是矩阵指数的存储只能存储Ta，而不是全量。因为Ta很小，当其与单位矩阵相加时候，其精度将丧失殆尽。这是计算数学中的“大数吃小数现象”，需要特别注意的。对矩阵做分解

$$\mathbf{T} = (\mathbf{I}_n + \mathbf{T}_a)^{2N} = (\mathbf{I}_n + \mathbf{T}_a)^{2(N-1)} \times (\mathbf{I}_n + \mathbf{T}_a)^{2(N-1)}$$

在矩阵计算中Ib，Ic，有

$$(\mathbf{I} + \mathbf{T}_b) \times (\mathbf{I} + \mathbf{T}_c) = \mathbf{I} + \mathbf{T}_b + \mathbf{T}_c + \mathbf{T}_b \times \mathbf{T}_c$$

精细积分法

当Ib， Ic很小时，不应该加上单位矩阵后再执行乘法，而是将Ib， Ic都看成Ia。

在做完矩阵N次分解之后，再执行

$$\mathbf{T} = \mathbf{I}_b + \mathbf{T}_a$$

此时Ta已经不是小量，没有严重的舍入误差了，以上便是精细积分法的基本思想与完整过程。

精细积分范例

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\frac{1}{10} & \frac{1}{40} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix} \mathbf{x}, \mathbf{x}_0 = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 30\exp\left(\frac{-3t}{20}\right) + 30\exp\left(\frac{-t}{20}\right) \\ -60\exp\left(\frac{-3t}{20}\right) + 60\exp\left(\frac{-t}{20}\right) \end{bmatrix}$$

	精细积分		解析结果	
t	x(2)	x(2)	x(1)	x(2)
0.5	57.091602	2.853986	57.091602	2.853986
1.0	54.358122	5.431287	54.358122	5.431287
1.5	51.787791	7.753636	51.787791	7.753636
2.0	49.369669	9.841152	49.369669	9.841152
2.5	47.093585	11.712457	47.093585	11.712457
3.0	44.950084	13.384789	44.950084	13.384789
3.5	42.930372	14.874099	42.930372	14.874099
4.0	41.026272	16.195147	41.026272	16.195147
4.5	39.230179	17.361588	39.230179	17.361588
5.0	37.535020	18.386054	37.535020	18.386054
5.5	35.934213	19.280228	35.934213	19.280228
6.0	34.421636	20.054914	34.421636	20.054914
6.5	[illegible]	[illegible]	[illegible]	[illegible]

作业：

编写四阶Adams预测校正方法积分器，计算以下初值问题。

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

$\mu = 398{,}600.4415 \pm 0.0008\ km^3/s^2$

积分步长1分钟，要求积分3小时，输出积分文件，并画出空间三维图形。初始状态为
[6678.137, 0, 0, 0, 6.789530 , 3.686414]
单位km,km/s。

作业提交时间：2周内。

低阶Adams预测校正方法

四阶显式Adams公式与三阶隐式公式联合使用。

预测公式为：

$$\boldsymbol{y}^{(0)}{}_{n+1} = \boldsymbol{y}_n + \frac{h}{24}\left[55\boldsymbol{f}_n - 59\boldsymbol{f}_{n-1} + 37\boldsymbol{f}_{n-2} - 9\boldsymbol{f}_{n-3}\right]$$

校正公式为：

$$\boldsymbol{y}_{n+1}^{(i+1)} = \boldsymbol{y}_n + \frac{h}{24}[9\boldsymbol{f}_{n+1}^{(i)} + 19\boldsymbol{f}_n - 5\boldsymbol{f}_{n-1} + \boldsymbol{f}_{n-2}]$$

其 PECE 模式为：

P 步:

$$\boldsymbol{y}^{(0)}{}_{n+1} = \boldsymbol{y}_n + \frac{h}{24}\left[55\boldsymbol{f}_n - 59\boldsymbol{f}_{n-1} + 37\boldsymbol{f}_{n-2} - 9\boldsymbol{f}_{n-3}\right]$$

E 步:

$$\boldsymbol{f}_{n+1}^{(0)} = \boldsymbol{f}\left(t_{n+1}, \boldsymbol{y}^{(0)}{}_{n+1}\right)$$

C 步:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{h}{24}[9\boldsymbol{f}_{n+1}^{(0)} + 19\boldsymbol{f}_n - 5\boldsymbol{f}_{n-1} + \boldsymbol{f}_{n-2}]$$

E 步:

$$\boldsymbol{f}_{n+1} = \boldsymbol{f}\left(t_{n+1}, \boldsymbol{y}_{n+1}\right)$$

Q&A!