中国科学院上海天文台
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势函数与引力场模型

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2019年秋季

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课件地址：http://202.127.29.4/astrodynamics/course.php

课程主要内容

- 引力势函数
- 分离变量法
- 勒让德多项式球谐函数
- 引力位低阶展开式
- 引力场模型

单位质量引力及空间积分

单位质量力“元”

$$\delta\mathbf{F}(\mathbf{x}) = Gm_{\mathrm{s}}\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\delta m(\mathbf{x}') = Gm_{\mathrm{s}}\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}')\,\mathrm{d}^3\mathbf{x}'$$

力元对空间积分

$$\mathbf{g}(\mathbf{x}) \equiv G\int \mathrm{d}^3\mathbf{x}'\,\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}')$$

$$\mathbf{F}(\mathbf{x}) = m_{\mathrm{s}}\mathbf{g}(\mathbf{x})$$

引力势标量场

定义引力势

$$\Phi(\mathbf{x}) \equiv -G \int \mathrm{d}^3\mathbf{x}' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|},$$

注意到

$$\boldsymbol{\nabla}_{\mathbf{x}} \left(\frac{1}{|\mathbf{x}' - \mathbf{x}|} \right) = \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3}$$

则

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &= \boldsymbol{\nabla}_{\mathbf{x}} \int \mathrm{d}^3\mathbf{x}' \, \frac{G\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \\ &= -\boldsymbol{\nabla}\Phi, \end{aligned}$$

泊松方程

对g(x)求散度

$$\nabla \cdot \mathbf{g}(\mathbf{x}) = G \int \mathrm{d}^3\mathbf{x}'\, \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \rho(\mathbf{x}').$$

$$\nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = -\frac{3}{|\mathbf{x}' - \mathbf{x}|^3} + \frac{3(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^5}$$

$$\begin{aligned}
\nabla \cdot \mathbf{g}(\mathbf{x}) &= G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| \le h} \mathrm{d}^3\mathbf{x}'\, \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \\
&= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| \le h} \mathrm{d}^3\mathbf{x}'\, \nabla_{\mathbf{x}'} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \\
&= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}| = h} \mathrm{d}^2\mathbf{S}' \cdot \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3}.
\end{aligned}$$

泊松方程与拉普拉斯方程

$$\boldsymbol{\nabla}\cdot\mathbf{g}(\mathbf{x}) = -G\rho(\mathbf{x})\int \mathrm{d}^2\Omega = -4\pi G\rho(\mathbf{x})$$

带入势函的梯度，得到泊松方程

$$\nabla^2\Phi = 4\pi G\rho.$$

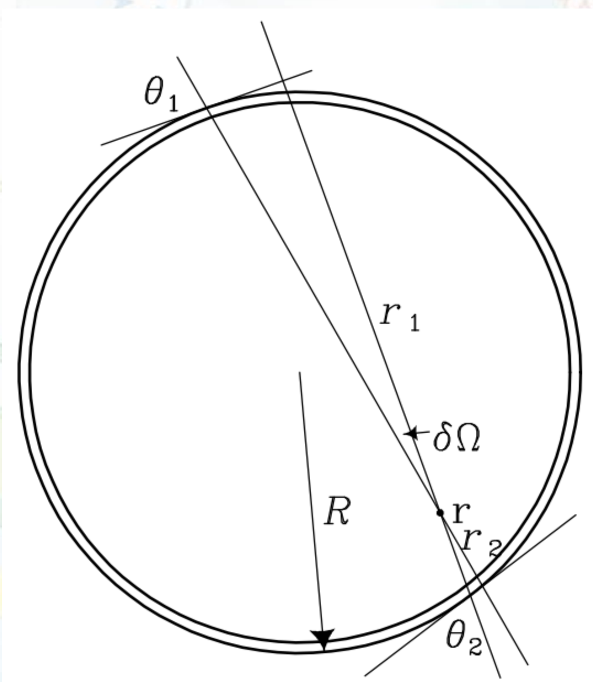
泊松方程提供了求解引力势的一条途径，然后再通过求解梯度得到引力，这往往比力元进行积分更为方便。

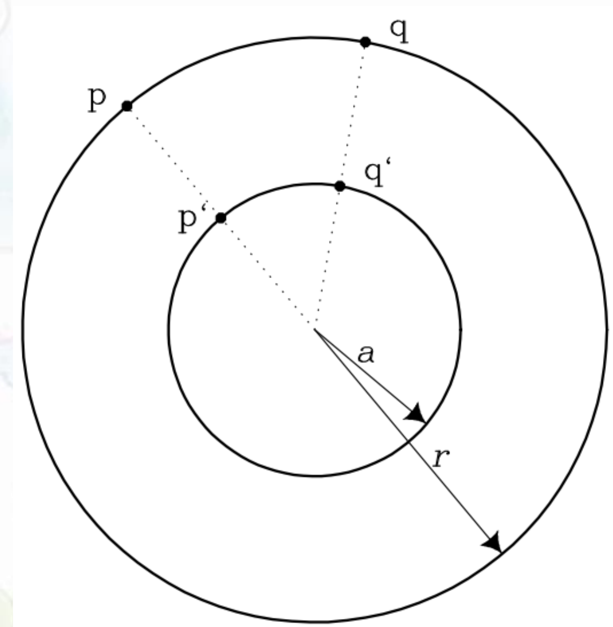
对于密度为0的情况，为拉普拉斯方程

$$\nabla^2\Phi = 0$$

牛顿定理

- 牛顿第一定理：球壳内位于其内部任意一点上的物理的引力之和为零。
- 牛顿第二定理：闭合球壳对位于球壳外任一物体的引力，等于把球壳所有质量集中于球壳中心上的点质量对该物体的引力。





正交曲线坐标系中拉普拉斯算子

$$\nabla^2 F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial F}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial F}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial F}{\partial q_3}\right)\right]$$

$$\boldsymbol{e}_i = \frac{1}{H_i}\frac{\partial \boldsymbol{r}}{\partial q_i}, \qquad \frac{\partial \boldsymbol{r}}{\partial q_i} = \frac{\partial x}{\partial q_i}\boldsymbol{i} + \frac{\partial y}{\partial q_i}\boldsymbol{j} + \frac{\partial z}{\partial q_i}\boldsymbol{k}$$

$$H_i = \left|\frac{\partial \boldsymbol{r}}{\partial q_i}\right| = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}$$

二体问题中曾经定义过拉梅系数

引力势的多极展开

如果球壳的厚度忽略不计，求解泊松方程的任务简化为求解球壳内部和外部的拉普拉斯方程。球坐标系下拉普拉斯方程为

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\nabla^2 F=\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2};$$

笛卡尔坐标

$$\nabla^2 F=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial F}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 F}{\partial\phi^2}+\frac{\partial^2 F}{\partial z^2}$$

柱坐标

分离变量

$$\Phi(r,\theta,\phi) = R(r)P(\theta)Q(\phi)$$

带入拉普拉斯方程

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{\sin\theta}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right) = -\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2}$$

左边与phi无关，右边与r,theta无关，
只能两者都等于常数，如m^2

$$-\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2} = m^2$$

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{\sin\theta}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right) = m^2$$

常微分方程求解

$$Q(\phi) = Q_m^+ \mathrm{e}^{im\phi} + Q_m^- \mathrm{e}^{-im\phi}$$

第一式积分结果

$$Q = Q_m \mathrm{e}^{im\phi} \qquad (m = \ldots, -1, 0, 1, \ldots).$$

第二式变形

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \frac{m^2}{\sin^2\theta} - \frac{1}{P\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)$$

左端与theta无关，右端与r无关，则只能两者等于常数，记为l(l+1)

r方程的线性独立解

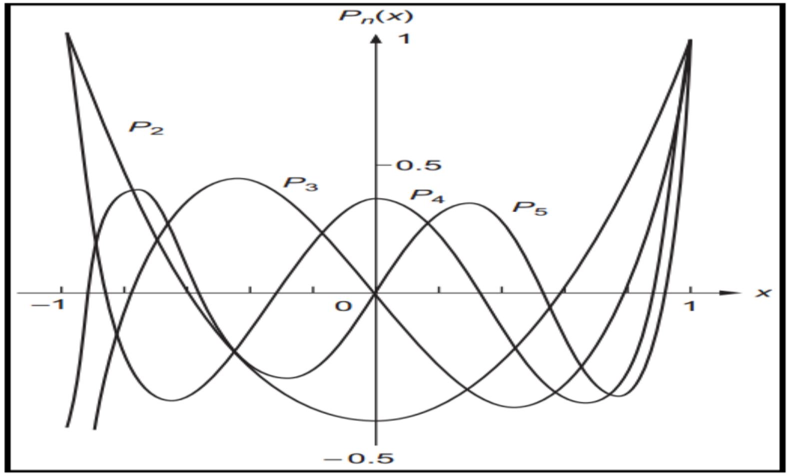
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - l(l+1)R = 0$$

$$R(r) = Ar^l \quad \text{and} \quad R(r) = Br^{-(l+1)}$$

勒让德函数

$x \equiv \cos\theta$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right]-\frac{m^2}{1-x^2}P+l(l+1)P=0$$



Legendre polynomials $P_2(x)$ through $P_5(x)$.

生成函数

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l \quad |t|<1,\ |x|\le 1$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

$$r_< = \min(|\mathbf{x}|, |\mathbf{x}'|),\ r_> = \max(|\mathbf{x}|, |\mathbf{x}'|)$$

递推关系

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad n = 1, 2, 3, \ldots.$$

$$2P_2(x) = 3xP_1(x) - P_0(x) \quad \longrightarrow \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right).$$

$$\begin{aligned}
P'_{n+1}(x) &= (n+1)P_n(x) + xP'_n(x), \\
P'_{n-1}(x) &= -nP_n(x) + xP'_n(x), \\
(1-x^2)P'_n(x) &= nP_{n-1}(x) - nxP_n(x), \\
(1-x^2)P'_n(x) &= (n+1)xP_n(x) - (n+1)P_{n+1}(x).
\end{aligned}$$

低阶多项式

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$P_7(x) = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

$$P_8(x) = \tfrac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$$

勒让德级数对函数的逼近

$$\psi(r,\theta)=\sum_{l=0}^{\infty}(a_l r^l + b_l r^{-l-1})P_l(\cos\theta).$$

$$\psi(r,\theta)=\sum_{l=0}^{\infty}a_l\, r^l P_l(\cos\theta) \quad (r\le r_0),$$

$$\psi(r,\theta)=\sum_{l=0}^{\infty}a_l\, r^{-l-1} P_l(\cos\theta) \quad (r\ge r_0).$$

$$U(r,\theta)=\frac{GM}{R}\left[\frac{R}{r}-\sum_{l=2}^{\infty}a_l\left(\frac{R}{r}\right)^{l+1}P_l(\cos\theta)\right].$$

缔合勒让德多项式

第一类与第二类缔合勒让德函数是下面微分方程的线性独立解。

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}w}{\mathrm{d}z}\right]-\frac{\mu^2}{1-z^2}w+\lambda(\lambda+1)w=0.$$

$$P_\lambda^\mu(x)\equiv\frac{1}{2}\lim_{\epsilon\to 0}\left[\mathrm{e}^{\pi\mathrm{i}\mu/2}P_\lambda^\mu(x+\mathrm{i}|\epsilon|)+\mathrm{e}^{-\pi\mathrm{i}\mu/2}P_\lambda^\mu(x-\mathrm{i}|\epsilon|)\right]$$

$$Q_\lambda^\mu(x)\equiv\frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\mu}\lim_{\epsilon\to 0}\left[\mathrm{e}^{-\pi\mathrm{i}\mu/2}Q_\lambda^\mu(x+\mathrm{i}|\epsilon|)+\mathrm{e}^{\pi\mathrm{i}\mu/2}Q_\lambda^\mu(x-\mathrm{i}|\epsilon|)\right]$$

$$P_l(x)\equiv P_l^0(z)=\frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l$$

正交性

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{\mathrm{d}^m P_l^0(x)}{\mathrm{d}x^m} = (-1)^m \frac{(1-x^2)^{m/2}}{2^l l!} \frac{\mathrm{d}^{l+m} P_l^0(x)}{\mathrm{d}x^{l+m}}$$

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

$$\int_{-1}^{1} \mathrm{d}x\, P_l^m(x) P_n^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ln}.$$

$$\int_{-1}^{1} \frac{\mathrm{d}x}{1-x^2} P_l^m(x) P_l^k(x) = \frac{1}{m} \frac{(l+m)!}{(l-m)!} \delta_{mk}.$$

低阶缔合勒让德多项

$$P_1^1(x) = -(1-x^2)^{1/2} = -\sin\theta$$

$$P_2^1(x) = -3x(1-x^2)^{1/2} = -3\cos\theta\sin\theta$$

$$P_2^2(x) = 3(1-x^2) = 3\sin^2\theta$$

$$P_3^1(x) = -\tfrac{3}{2}(5x^2-1)(1-x^2)^{1/2} = -\tfrac{3}{2}(5\cos^2\theta-1)\sin\theta$$

$$P_3^2(x) = 15x(1-x^2) = 15\cos\theta\sin^2\theta$$

$$P_3^3(x) = -15(1-x^2)^{3/2} = -15\sin^3\theta$$

$$P_4^1(x) = -\tfrac{5}{2}(7x^3-3x)(1-x^2)^{1/2} = -\tfrac{5}{2}(7\cos^3\theta-3\cos\theta)\sin\theta$$

$$P_4^2(x) = \tfrac{15}{2}(7x^2-1)(1-x^2) = \tfrac{15}{2}(7\cos^2\theta-1)\sin^2\theta$$

$$P_4^3(x) = -105x(1-x^2)^{3/2} = -105\cos\theta\sin^3\theta$$

$$P_4^4(x) = 105(1-x^2)^2 = 105\sin^4\theta$$

系数归一化

Geopotential models are generally expressed in terms of normalised coefficients C^*_{lm}, while the coefficients C_{lm} used above are referred to as non-normalised. The relation between C^*_{lm} and C_{lm} is

$$C^*_{lm} = \sqrt{\frac{(l+m)!}{(l-m)!(2l+1)(2-\delta_{0m})}} C_{lm}$$

where δ_{0m} is the Kronecker symbol, equal to 1 if $m = 0$ or 0 if $m \neq 0$

$$C_{lm} = N_{lm} \bar{C}_{lm}$$
$$S_{lm} = N_{lm} \bar{S}_{lm}$$
$$N_{lm} = \sqrt{\frac{(l-m)!(2l+1)(2-\delta_{0m})}{(l+m)!}}$$

归一化勒让德多项式递推方法

$$
\begin{cases}
\bar{P}_{1,1}(u) = \sqrt{3(1-u^2)} \\
\bar{P}_{l,l}(u) = \sqrt{\dfrac{2l+1}{2l}}\sqrt{1-u^2}\,\bar{P}_{l-1,l-1}(u) \\
\bar{P}_{l,m}(u) = \sqrt{\dfrac{(2l+1)(2l-1)}{(l+m)(l-m)}}\,u\bar{P}_{l-1,m}(u) \\
\qquad -\sqrt{\dfrac{(2l+1)(l-1+m)(l-1-m)}{(2l-3)(l+m)(l-m)}}\,\bar{P}_{l-2,m}(u) \\
l \geq 2, m = 1, 2, \cdots, l-1 \\
\bar{P}_{i,j}(u) = 0, i < j
\end{cases}
$$

$$
\bar{P}_l(u) = \sqrt{\frac{2l+1}{2l-1}}\left[\left(1-\frac{1}{l}\right)u\bar{P}_{l-1}(u) - \sqrt{\frac{2l-1}{2l-3}}\left(1-\frac{1}{l}\right)\bar{P}_{l-2}(u)\right], l \geq 2
$$

$$
\bar{P}_0(u) = 1, \bar{P}_1(u) = \sqrt{3}u
$$

不同递推方法，对累积误差损失不同，这里给出的方法较传统方法精度损失较小，推荐使用。

详见： S.A. Holmes Jounal of Geodesy 2002

A unitfied approach to the Clenshaw summation and recursive computation of very high degree and order normalised associated Legendre functions

归一化勒让德多项式计算

```
subroutine Legendre(PG,PZ,N,u)
!---------------------------------subroutine  comment
!  Purpose    :  递推法计算勒让德多项式带谐项和田谐项
!  Author     :  Song Yezhi       <song.yz@foxmail.com>
!  Versions and Changes   :
!       V1.0 ----2014-02-17 15:11:24
!           递推法计算正交归一化勒让德多项式
!           算法详见《航天器轨道理论》
!           田谐项和带谐项解耦
!       V2.0 ----2014-02-27 17:10:28
!           递推过程参考V1.0公式，与以下公式同
!            S.A. Holmes   Jounal of Geodesy  2002
!            A unitfied approach to the Clenshaw summation and recursive
!            computation of very high degree and order normalised associated
!            Legendre functions
!-----------------------------------------------------
!  Input Parameters     :
!       N------------阶数
!       u------------自变量
!     注意:
!       u-----------在势函数理论中（如牛顿引力势、静电场势等）
!                        u-实际为  sin(phi)
!                        phi为展开点在球坐标下的纬度
!  Output Parameters    :
!       PG(N,N)  -------归一化勒让德多项式田谐项
!       PZ(0:N)---------归一化勒让德多项式带谐项
!     注意:
!        PZ的指标从0阶开始，0阶为常数项
!        PG的指标从1阶开始
!  Subroutines used     :
!       *. geo_legendre(PG,N,u)
!       *. zone_legendre(PZ,N,u)
!-----------------------------------------------------
!  Copyrigt (C)  :
!        Center for Astro-geodynamics
!        Shanghai Astronomical Observatory
!----------------------------------------------------
implicit real*8(a-h,o-z)
real*8::PG(N,N),PZ(0:N)
 call  geo_legendre(PG,N,u)
 call  zone_legendre(PZ,N,u)
 contains
```

归一化勒让德多项式计算

```fortran
subroutine  geo_legendre(PG,N,u)
!---------------------------------subroutine  comment
!  Purpose    :  计算归一化勒让德多项式田谐项
!  Author     :  song.yz
!  Created    :  2014-02-10 20:06:19
!-----------------------------------------------------
!  Input Parameters     :
!       N------阶数
!       u------自变量
!  Output Parameters    :
!         PG（N,N） 归一化勒让德多项式
!-----------------------------------------------------
implicit real*8(a-h,o-z)
real*8::PG(N,N)
PG=0D0
PG(1,1)=dsqrt(3*(1-u*u))
!计算所有扇谐项
do L=2,N
      TMP=(2d0*L+1)/2D0/L*(1-u*u)
      TMP=dsqrt(TMP)
      PG(L,L)=TMP*PG(L-1,L-1)
end do
!从第二列开始计算, 至倒数第二列, 最后一列为扇谐项已经计算完毕
!为了提高计算效率, 第一列暂不计算, 对角线以上的元素为0, 前面已经设置完毕
do M=2,N-1
  do L=M+1,N
   tmp1=(2d0*L+1D0)*(2D0*L-1D0)/(L+M)/(L-M)
   tmp1=dsqrt(tmp1)*u
   tmp2=(2d0*L+1)*(L-1D0+M)*(L-1D0-M)/(2D0*L-3D0)/(L+M)/(L-M)
   tmp2=dsqrt(tmp2)
   PG(L,M)=tmp1*PG(L-1,M)-tmp2*PG(L-2,M)
   end do
end do
! 计算第一列
M=1
do L=2,N
   tmp1=(2d0*L+1D0)*(2D0*L-1D0)/(L+M)/(L-M)
   tmp1=dsqrt(tmp1)*u
   tmp2=(2d0*L+1)*(L-1D0+M)*(L-1D0-M)/(2D0*L-3D0)/(L+M)/(L-M)
   tmp2=dsqrt(tmp2)
   if (L==2) then
   PG(L,M)=tmp1*PG(L-1,M)
   else
    PG(L,M)=tmp1*PG(L-1,M)-tmp2*PG(L-2,M)
   end if
end do
end subroutine geo_legendre
```

归一化勒让德多项式计算

```fortran
subroutine zone_legendre(PZ,N,u)
!---------------------------------subroutine  comment
!  Copyright     :  Shanghai Astronomical Observatory
!  Author        :  song.yz
!  Purpose       :
!  Version & Changes    :
!      V 1.0- (Created)---------- 2014-01-14 15:31:46
!         归一化Legendre带谐项多项式计算函数
!         《航天器轨道理论》---P109
!         注意: 带谐项从0开始, 这样可以直接与球谐展开中的常数项合并
!               而球谐项则从1开始, 如果球谐项从0的话, 则其中有一项是冗余的
!-----------------------------------------------------
!  Input Parameters     :
!           n-------------最大带谐阶数
!           u-------------自变量
!  Output Parameters    :
!           PZ(0:N)----------归一化legendre带谐项函数值
!----------------------------------------------------
implicit real*8(a-h,o-z)
real*8::PZ(0:N)
PZ(0)=1D0
PZ(1)=dsqrt(3d0)*u
do i=2,N
    TMP1=(2D0-1D0/i)*u
    TMP2=dsqrt((2D0*i-1D0)/(2D0*i-3D0))*(1D0-1d0/i)
    TMP3=dsqrt((2D0*i+1)/(2d0*i-1))
    PZ(i)=TMP3*(TMP1*PZ(i-1)-TMP2*PZ(i-2))
end do
end subroutine  zone_legendre
end subroutine Legendre
```

球谐函数

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \qquad (m \geq 0)$$

$$\begin{aligned}\int d^2\Omega\, Y_l^{m*}(\boldsymbol{\Omega})Y_{l'}^{m'}(\boldsymbol{\Omega}) &\equiv \int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\phi\, Y_l^{m*}(\theta,\phi)Y_{l'}^{m'}(\theta,\phi) \\ &= \delta_{ll'}\delta_{mm'},\end{aligned}$$

内部势与外部势

$$\Phi_{lm}(r,\mathbf{\Omega}) = \left(A_{lm}r^{l} + B_{lm}r^{-(l+1)}\right) \mathrm{Y}_{l}^{m}(\mathbf{\Omega})$$

$$\Phi_{\text{int}}(r,\mathbf{\Omega}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^{l} + B_{lm}r^{-(l+1)}\right) \mathrm{Y}_{l}^{m}(\mathbf{\Omega}) \quad (r \le a)$$

$$\Phi_{\text{ext}}(r,\mathbf{\Omega}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(C_{lm}r^{l} + D_{lm}r^{-(l+1)}\right) \mathrm{Y}_{l}^{m}(\mathbf{\Omega}) \quad (r \ge a)$$

球谐函数

球谐展开

$$f(\mathbf{r}) = f(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(r)\mathrm{Y}_l^m(\theta,\phi)$$

勒让德多项式的球谐表示

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\mathrm{Y}_l^{m*}(\theta',\phi')\mathrm{Y}_l^m(\theta,\phi)$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\mathrm{Y}_l^{m*}(\theta',\phi')\mathrm{Y}_l^m(\theta,\phi),$$

低阶球谐项

$$Y_0^0(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$$

$$Y_1^1(\theta,\varphi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}=-\sqrt{\frac{3}{8\pi}}\,(x+iy)/r$$

$$Y_1^0(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta=\sqrt{\frac{3}{4\pi}}\,z/r$$

$$Y_1^{-1}(\theta,\varphi)=+\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}=\sqrt{\frac{3}{8\pi}}\,(x-iy)/r$$

$$Y_2^2(\theta,\varphi)=\sqrt{\frac{5}{96\pi}}\,3\sin^2\theta\, e^{2i\varphi}=3\sqrt{\frac{5}{96\pi}}\,(x^2-y^2+2ixy)/r^2$$

$$Y_2^1(\theta,\varphi)=-\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta\, e^{i\varphi}=-\sqrt{\frac{5}{24\pi}}\,3z(x+iy)/r^2$$

$$Y_2^0(\theta,\varphi)=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}z^2-\frac{1}{2}r^2\right)/r^2$$

$$Y_2^{-1}(\theta,\varphi)=\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta\, e^{-i\varphi}=+\sqrt{\frac{5}{24\pi}}\,3z(x-iy)/r^2$$

$$Y_2^{-2}(\theta,\varphi)=\sqrt{\frac{5}{96\pi}}\,3\sin^2\theta\, e^{-2i\varphi}=3\sqrt{\frac{5}{96\pi}}\,(x^2-y^2-2ixy)/r^2$$

$$Y_3^3(\theta,\varphi)=-\sqrt{\frac{7}{2880\pi}}\,15\sin^3\theta\, e^{3i\varphi}=-\sqrt{\frac{7}{2880\pi}}\,15[x^3-3xy^2+i(3x^2y-y^3)]/r^3$$

$$Y_3^2(\theta,\varphi)=\sqrt{\frac{7}{480\pi}}\,15\cos\theta\sin^2\theta\, e^{2i\varphi}=\sqrt{\frac{7}{480\pi}}\,15z(x^2-y^2+2ixy)/r^3$$

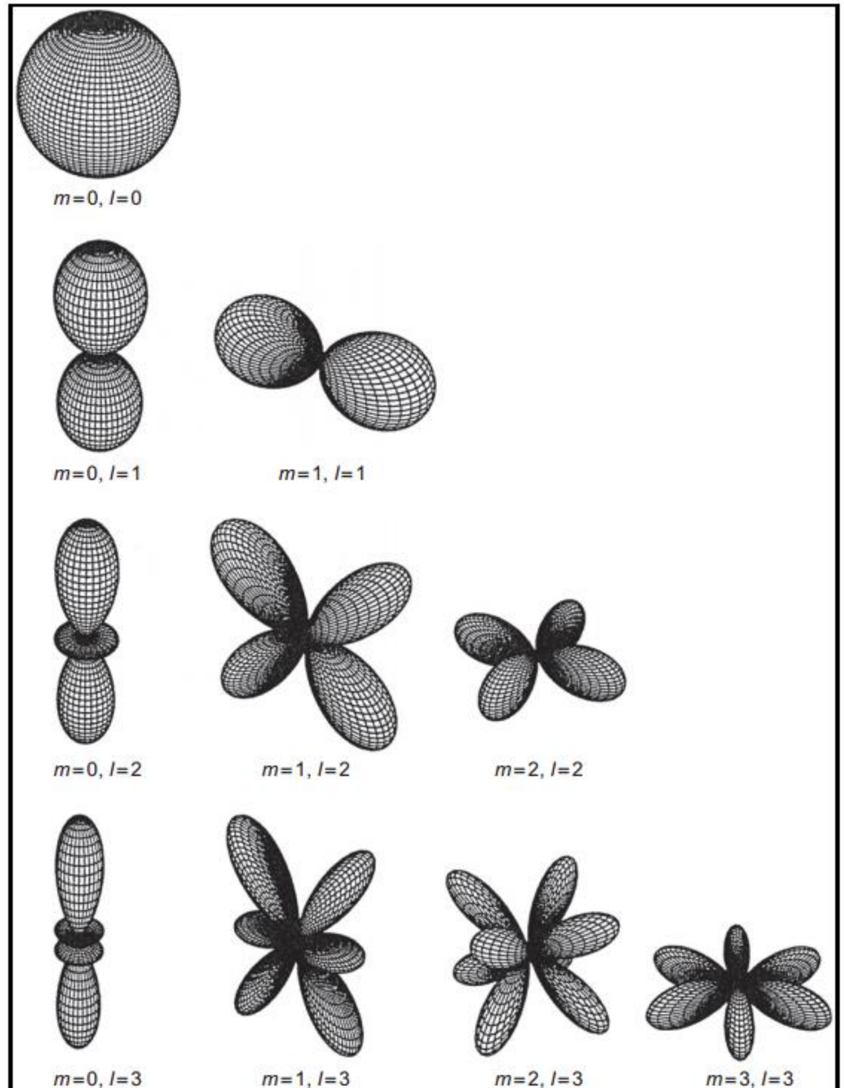
$$Y_3^1(\theta,\varphi)=-\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}cos^2\theta-\frac{3}{2}\right)\sin\theta\, e^{i\varphi}=-\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}z^2-\frac{3}{2}r^2\right)(x+iy)/r^3$$

$$Y_3^0(\theta,\varphi)=\sqrt{\frac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)=\sqrt{\frac{7}{4\pi}}\,z\left(\frac{5}{2}z^2-\frac{3}{2}r^2\right)/r^3$$

$$Y_3^{-1}(\theta,\varphi)=+\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}cos^2\theta-\frac{3}{2}\right)\sin\theta\, e^{-i\varphi}=\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}z^2-\frac{3}{2}r^2\right)(x-iy)/r^3$$

$$Y_3^{-2}(\theta,\varphi)=\sqrt{\frac{7}{480\pi}}\,15\cos\theta\sin^2\theta\, e^{-2i\varphi}=\sqrt{\frac{7}{480\pi}}\,15z(x^2-y^2-2ixy)/r^3$$

$$Y_3^{-3}(\theta,\varphi)=+\sqrt{\frac{7}{2880\pi}}\,15\sin^3\theta\, e^{-3i\varphi}=\sqrt{\frac{7}{2880\pi}}\,15[x^3-3xy^2-i(3x^2y-y^3)]/r^3$$



Shapes of $|\Re e Y_l^m(\theta,\varphi)|^2$ for $0\le l\le 3$, $m=0\ldots l$.

拉普拉斯级数-引力场球谐展开

$$U(r,\theta,\varphi)=\frac{GM}{R}\left[\frac{R}{r}-\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{R}{r}\right)^{l+1}\left[C_{lm}Y_{ml}^{e}(\theta,\varphi)+S_{lm}Y_{ml}^{o}(\theta,\varphi)\right]\right]$$

$$Y_{ml}^{e}(\theta,\varphi)=P_l^m(\cos\theta)\cos m\varphi,\quad Y_{ml}^{o}(\theta,\varphi)=P_l^m(\cos\theta)\sin m\varphi.$$

Gravity Field Coefficients

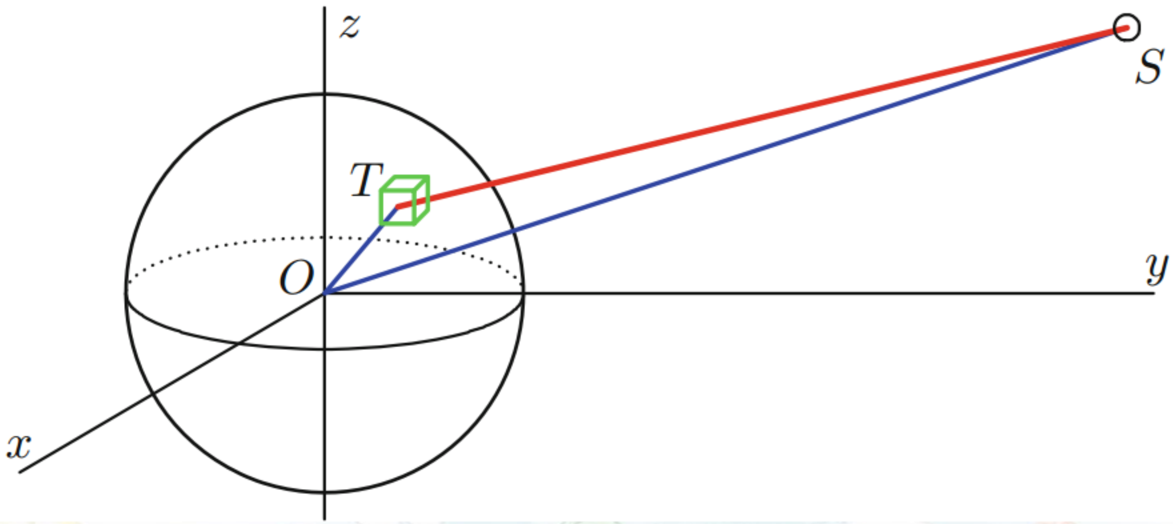
Coefficient[a]	Earth	Moon	Mars
C_{20}	1.083×10^{-3}	$(0.200\pm0.002)\times10^{-3}$	$(1.96\pm0.01)\times10^{-3}$
C_{22}	0.16×10^{-5}	$(2.4\pm0.5)\times10^{-5}$	$(-5\pm1)\times10^{-5}$
S_{22}	-0.09×10^{-5}	$(0.5\pm0.6)\times10^{-5}$	$(3\pm1)\times10^{-5}$

球谐函数递推公式

$$\cos\theta\, Y_l^m = \left[\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right]^{1/2} Y_{l+1}^m + \left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2} Y_{l-1}^m,$$

$$e^{\pm i\varphi}\sin\theta\, Y_l^m = \mp\left[\frac{(l\pm m+1)(l\pm m+2)}{(2l+1)(2l+3)}\right]^{1/2} Y_{l+1}^{m\pm 1} \pm \left[\frac{(l\mp m)(l\mp m-1)}{(2l-1)(2l+1)}\right]^{1/2} Y_{l-1}^{m\pm 1}.$$

引力位元



$$\frac{\boldsymbol{OT}}{\rho} = \begin{pmatrix} \cos\beta\cos\alpha \\ \cos\beta\sin\alpha \\ \sin\beta \end{pmatrix}$$

$$\frac{\boldsymbol{OS}}{r} = \begin{pmatrix} \cos\psi\cos\lambda \\ \cos\psi\sin\lambda \\ \sin\psi \end{pmatrix}$$

$$D^2 = \|\boldsymbol{TS}\|^2 = r^2 - 2r\rho\cos\theta + \rho^2$$

$$D = D(T,S) = r\left[1 - 2\frac{\rho}{r}\cos\theta + \left(\frac{\rho}{r}\right)^2\right]^{1/2}$$

$$\cos\theta = \sin\psi\sin\beta + \cos\psi\cos\beta\cos(\lambda - \alpha)$$

$$\mathrm{d}U = \frac{\mathrm{d}\mu}{D} = G\frac{\mathrm{d}M}{D}$$

通过积分计算引力位

$$U = U(S) = \int_{\mathrm{Earth}} \mathrm{d}U = \mathcal{G} \int_{T \in \mathrm{Earth}} \frac{\mathrm{d}M(T)}{D(T,S)}$$

$$\frac{1}{D} = \frac{1}{r} \frac{1}{\sqrt{1 - 2\frac{\rho}{r}\cos\theta + \left(\frac{\rho}{r}\right)^2}}$$

$$\frac{1}{D} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{\rho}{r}\right)^l P_l(\cos\theta)$$

加法公式

$$P_l(\cos\theta) = P_l(\sin\psi) \cdot P_l(\sin\beta) + 2\sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_{lm}(\sin\psi) P_{lm}(\sin\beta) \cos m(\lambda - \alpha)$$

通过积分计算引力位

$$U(r,\lambda,\psi) = G\int_\rho\int_\alpha\int_\beta \frac{\mathrm{d}M(\rho,\alpha,\beta)}{D(r,\lambda,\psi,\rho,\alpha,\beta)}$$

$$= G\frac{1}{r}\int_{\rho=0}^{R}\int_{\alpha=0}^{2\pi}\int_{\beta=-\pi/2}^{\pi/2}\sum_{l=0}^{\infty}\left(\frac{\rho}{r}\right)^l\Bigg[P_l(\sin\psi)P_l(\sin\beta)$$

$$+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_{lm}(\sin\psi)\cos m\lambda P_{lm}(\sin\beta)\cos m\alpha$$

$$+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_{lm}(\sin\psi)\sin m\lambda P_{lm}(\sin\beta)\sin m\alpha\Bigg]\mathrm{d}M$$

$$U(r,\lambda,\psi) = \frac{\mu}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l\left[\sum_{m=0}^{l}\Big(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda\Big)P_{lm}(\sin\psi)\right]$$

$$M = \int_{\rho=0}^{R}\int_{\alpha=0}^{2\pi}\int_{\beta=-\pi/2}^{\pi/2}\mathrm{d}M(\rho,\alpha,\beta)$$

带谐项与球谐项系数

$$C_{l0} = \frac{1}{MR^l} \int_{\rho=0}^{R} \int_{\alpha=0}^{2\pi} \int_{\beta=-\pi/2}^{\pi/2} \rho^l P_l(\sin\beta) \mathrm{d}M(\rho, \alpha, \beta)$$

$$S_{l0} = 0\ .$$

$$C_{lm} = \frac{2}{MR^l} \frac{(l-m)!}{(l+m)!} \int_\rho \int_\alpha \int_\beta \rho^l P_{lm}(\sin\beta) \cos m\alpha \, \mathrm{d}M$$

$$S_{lm} = \frac{2}{MR^l} \frac{(l-m)!}{(l+m)!} \int_\rho \int_\alpha \int_\beta \rho^l P_{lm}(\sin\beta) \sin m\alpha \, \mathrm{d}M$$

低阶展开式（二阶）

$$U(r,\lambda,\psi) = \frac{\mu}{r}\Bigg\{C_{00}P_0(\sin\psi)$$

$$+\left(\frac{R}{r}\right)\Big[C_{10}P_1(\sin\psi) + (C_{11}\cos\lambda + S_{11}\sin\lambda)P_{11}(\sin\psi)\Big]$$

$$+\left(\frac{R}{r}\right)^2\Big[C_{20}P_2(\sin\psi) + (C_{21}\cos\lambda + S_{21}\sin\lambda)P_{21}(\sin\psi)$$

$$+(C_{22}\cos 2\lambda + S_{22}\sin 2\lambda)P_{22}(\sin\psi)\Big]\Bigg\}\ .$$

$$P_0(\sin\beta) = 1\ ,\quad P_1(\sin\beta) = \sin\beta\ ,\quad P_2(\sin\beta) = (3\sin^2\beta - 1)/2\ ,$$

$$P_{11}(\sin\beta) = \cos\beta\ ,\quad P_{21}(\sin\beta) = 3\sin\beta\cos\beta\ ,\quad P_{22}(\sin\beta) = 3\cos^2\beta$$

低阶系数

$$C_{00} = \frac{1}{M} \int_\rho \int_\alpha \int_\beta \mathrm{d}M(\rho, \alpha, \beta) = 1$$

$$C_{10} = \frac{1}{MR} \int_\rho \int_\alpha \int_\beta \rho \sin\beta \, \mathrm{d}M(\rho, \alpha, \beta)$$

$$= \frac{1}{MR} \iiint z \, \mathrm{d}M = \frac{z_0}{R}$$

$$C_{11} = \frac{1}{MR} \int_\rho \int_\alpha \int_\beta \rho \cos\beta \cos\alpha \, \mathrm{d}M(\rho, \alpha, \beta)$$

$$= \frac{1}{MR} \iiint x \, \mathrm{d}M = \frac{x_0}{R}$$

$$S_{11} = \frac{1}{MR} \int_\rho \int_\alpha \int_\beta \rho \cos\beta \sin\alpha \, \mathrm{d}M(\rho, \alpha, \beta)$$

$$= \frac{1}{MR} \iiint y \, \mathrm{d}M = \frac{y_0}{R}$$

低阶系数

$$
\begin{aligned}
C_{20} &= \frac{1}{MR^2}\int_\rho\int_\alpha\int_\beta \rho^2\frac{3\sin^2\beta-1}{2}\mathrm{d}M(\rho,\alpha,\beta)\\
&= \frac{1}{2MR^2}\iiint\left[3z^2-(x^2+y^2+z^2)\right]\mathrm{d}M\\
&= \frac{1}{2MR^2}\iiint\left[(x^2+z^2)+(y^2+z^2)-2(x^2+y^2)\right]\mathrm{d}M\\
&= \frac{1}{2MR^2}(I_x+I_y-2I_z)\\
C_{21} &= \frac{1}{3MR^2}\int_\rho\int_\alpha\int_\beta 3\rho^2\sin\beta\cos\beta\cos\alpha\,\mathrm{d}M(\rho,\alpha,\beta)\\
&= \frac{1}{MR^2}\iiint xz\,\mathrm{d}M=\frac{1}{MR^2}I_{xz}\\
S_{21} &= \frac{1}{3MR^2}\int_\rho\int_\alpha\int_\beta 3\rho^2\sin\beta\cos\beta\sin\alpha\,\mathrm{d}M(\rho,\alpha,\beta)\\
&= \frac{1}{MR^2}\iiint yz\,\mathrm{d}M=\frac{1}{MR^2}I_{yz}
\end{aligned}
$$

低阶系数

$$C_{21} = \frac{1}{3MR^2}\int_\rho\int_\alpha\int_\beta 3\rho^2 \sin\beta\cos\beta\cos\alpha\,\mathrm{d}M(\rho,\alpha,\beta)$$

$$= \frac{1}{MR^2}\iiint xz\,\mathrm{d}M = \frac{1}{MR^2}I_{xz}$$

$$S_{21} = \frac{1}{3MR^2}\int_\rho\int_\alpha\int_\beta 3\rho^2 \sin\beta\cos\beta\sin\alpha\,\mathrm{d}M(\rho,\alpha,\beta)$$

$$= \frac{1}{MR^2}\iiint yz\,\mathrm{d}M = \frac{1}{MR^2}I_{yz}$$

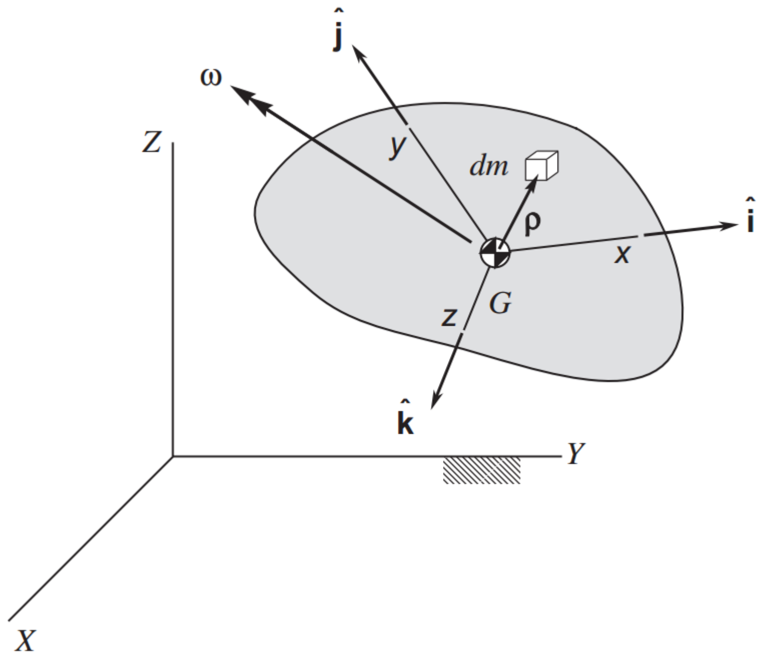
$$C_{22} = \frac{1}{12MR^2}\int_\rho\int_\alpha\int_\beta 3\rho^2 \cos^2\beta\cos 2\alpha\,\mathrm{d}M(\rho,\alpha,\beta)$$

$$= \frac{1}{4MR^2}\iiint (x^2 - y^2)\,\mathrm{d}M = \frac{1}{4MR^2}(I_y - I_x)$$

$$S_{22} = \frac{1}{12MR^2}\int_\rho\int_\alpha\int_\beta 3\rho^2 \cos^2\beta\sin 2\alpha\,\mathrm{d}M(\rho,\alpha,\beta)$$

$$= \frac{1}{2MR^2}\iiint xy\,\mathrm{d}M = \frac{1}{2MR^2}I_{xy}$$

低阶项与刚体惯量张量对应关系



$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$$I_x = \int (y^2 + z^2)\,dm \qquad I_{xy} = -\int xy\,dm \qquad I_{xz} = -\int xz\,dm$$

$$I_{yx} = -\int yx\,dm \qquad I_y = \int (x^2 + z^2)\,dm \qquad I_{yz} = -\int yz\,dm$$

$$I_{zx} = -\int zx\,dm \qquad I_{zy} = -\int zy\,dm \qquad I_z = \int (x^2 + y^2)\,dm$$

引力场模型

JGM-3 Earth Gravity Model

l m	$\overline{C}$	$\overline{S}$	$\sigma_{\overline{C}}$	$\sigma_{\overline{S}}$
2 0	−0.48416954845647E−03	0.00000000000000E+00	0.4660E−10	0.0000E+00
3 0	0.95717059088800E−06	0.00000000000000E+00	0.3599E−10	0.0000E+00
4 0	0.53977706835730E−06	0.00000000000000E+00	0.1339E−09	0.0000E+00
5 0	0.68658987986543E−07	0.00000000000000E+00	0.8579E−10	0.0000E+00
6 0	−0.14967156178604E−06	0.00000000000000E+00	0.2428E−09	0.0000E+00
7 0	0.90722941643232E−07	0.00000000000000E+00	0.2604E−09	0.0000E+00
8 0	0.49118003174734E−07	0.00000000000000E+00	0.3996E−09	0.0000E+00
2 1	−0.18698764000000E−09	0.11952801000000E−08	0.0000E+00	0.0000E+00
3 1	0.20301372055530E−05	0.24813079825561E−06	0.1153E−09	0.1152E−09
4 1	−0.53624355429851E−06	−0.47377237061597E−06	0.8693E−10	0.8734E−10
5 1	−0.62727369697705E−07	−0.94194632134383E−07	0.2466E−09	0.2465E−09
6 1	−0.76103580407274E−07	0.26899818932629E−07	0.2702E−09	0.2752E−09
7 1	0.28028652203689E−06	0.94777317813313E−07	0.4361E−09	0.4344E−09
8 1	0.23333751687204E−07	0.58499274939368E−07	0.5070E−09	0.5137E−09
2 2	0.24392607486563E−05	−0.14002663975880E−05	0.3655E−10	0.3709E−10
3 2	0.90470634127291E−06	−0.61892284647849E−06	0.9378E−10	0.9375E−10
4 2	0.35067015645938E−06	0.66257134594268E−06	0.1559E−09	0.1560E−09
5 2	0.65245910276353E−06	−0.32333435244435E−06	0.2392E−09	0.2398E−09
6 2	0.48327472124892E−07	−0.37381591944355E−06	0.3145E−09	0.3160E−09

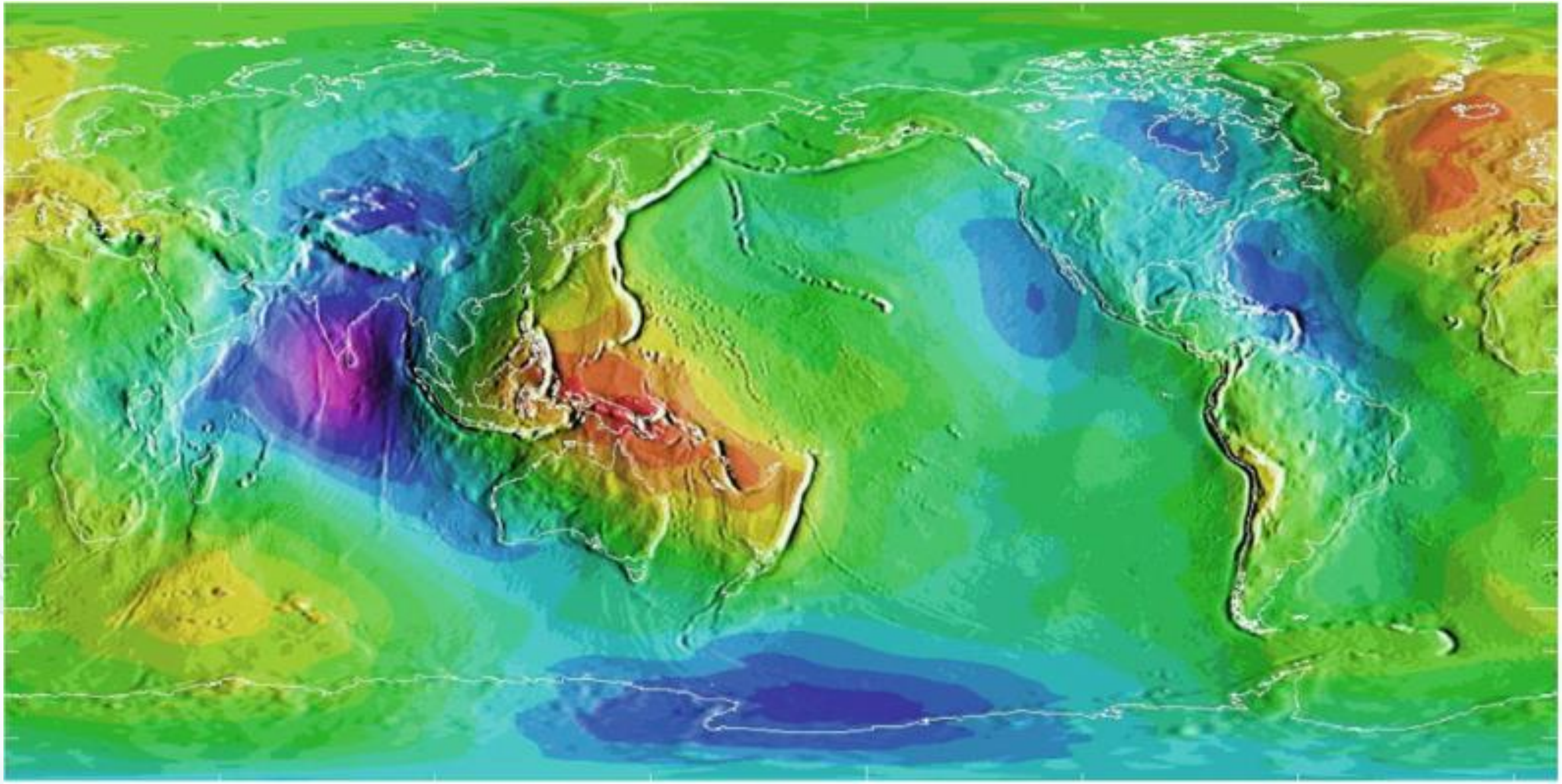
不同模型间比较

Coefficient	GEM-T2	JGM-3	GRIM5-C1	GRIM5-S1
C^*_{20}	−484.1652998	−484.165368	−484.16511551	−484.16511551
C^*_{30}	0.9570331	0.957171	0.95857491	0.95857492
C^*_{40}	0.5399078	0.539777	0.53978784	0.53978784
C^*_{50}	0.0686883	0.068659	0.06726760	0.06720440
C^*_{60}	−0.1496092	−0.149672	−0.14984936	−0.14985240
C^*_{70}	0.0900847	0.090723	0.09301877	0.09311367
C^*_{80}	0.0483835	0.049118	0.05039091	0.05046451
C^*_{90}	0.0284403	0.027385	0.02628356	0.02620763
C^*_{100}	0.0549673	0.054130	0.05101952	0.05076191
C^*_{200}	0.0199685	0.018790	0.02340848	0.02342817
C^*_{990}			−0.00128836	−0.00001554

Coefficient	EGM96	EIGEN-CH03S	EIGEN-6C2
C^*_{20}	−484.165371736	−484.165562843	−484.165299956
C^*_{30}	0.957254174	0.957477372	0.957208401
C^*_{40}	0.539873864	0.539923241	0.539990490
C^*_{50}	0.068532348	0.068584004	0.068684705
C^*_{60}	−0.149957995	−0.149991332	−0.149954200
C^*_{70}	0.090978937	0.090539419	0.090513612
C^*_{80}	0.049671167	0.049295631	0.049484115
C^*_{90}	0.027671430	0.028093014	0.028015031
C^*_{100}	0.052622249	0.053699211	0.053330869
C^*_{110}	−0.050961371	−0.050765723	−0.507685657
C^*_{120}	0.037725264	0.036209032	0.036437330
C^*_{130}	0.042298221	0.041543398	0.041729879
C^*_{140}	−0.024278650	−0.022288877	−0.022669657
C^*_{150}	0.001479101	0.002425544	0.002192288
C^*_{200}	0.022238461	0.021496270	0.021558749
C^*_{990}	0.001478118	−0.000779156	0.002263992
C^*_{22}	2.439143524	2.439311853	2.439355937
S^*_{22}	−1.400166837	−1.400342254	−1.400284583
C^*_{31}	2.029988822	2.030480649	2.030499314
S^*_{31}	0.248513159	0.248170920	0.248199233
C^*_{33}	0.721072657	0.721306788	0.721274250
S^*_{33}	1.414356270	1.414370341	1.414373139

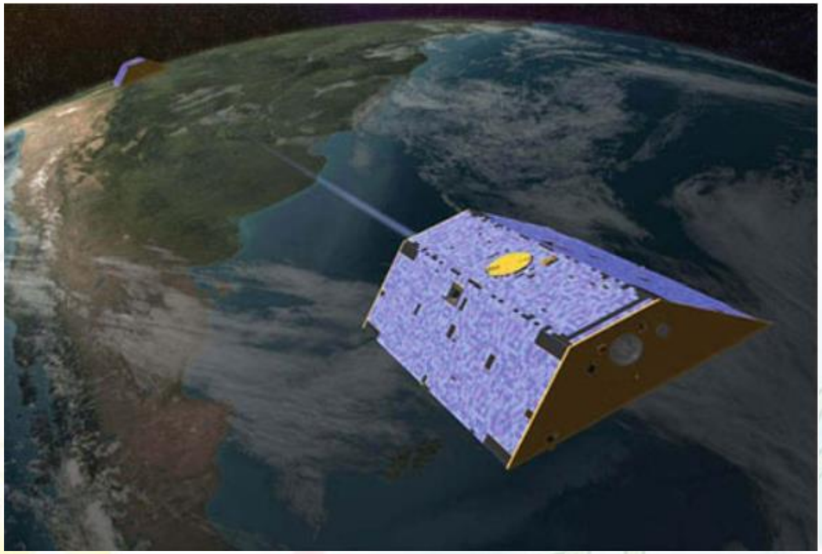
Comparison between different models. Normalised zonal coefficients Cl∗0 and other normalised coefficients Cl m ∗ and Sl m ∗ . All values should be multiplied
by 1E−6

大地水准面



Anomalies of the geoid in the EGM96 model . This map, using the platecarr´ee projection, represents the anomalies of the geoid (in meters) relative to the reference ellipsoid, as described by the US model EGM96.

卫星大地测量



The GRACE mission (Gravity Recovery And Climate Experiment) comprises two twin satellites GRACE-A and B, which follow one another around24 on the same orbit, separated by a distance of 200 km

The satellite GOCE (Gravity field and steady state Ocean Circulation Experiment) has improved our knowledge of the geoid still further, with a very low orbit (200 km) and a gradiometer comprising six accelerometers with an accuracy of 10-12 m s-2

主要引力场模型

- The first satellite data were integrated into existing models and, from 1970,certain models were established exclusively on the basis of space data. The SAO SE-1 model (Smithsonian Astrophysical Observatory-Standard Earth),considered to be the first satellite-only model, presented a degree 8 expansion of the geopotential in 1966. In 1973, the SE-3 model (degree 24) used the first laser ranging measurements to establish the distances to satellites.
- The GEM model (Goddard Earth Model) was established by NASA’s Goddard Space Flight Center (GSFC) in the United States as a reaction to the classified US military models. The first model GEM-1 was published in 1972,expanding the potential to degree 12. The GEM-T2 model, published in 1990,exploited the data from 31 satellites. It gave a model with all coefficients up to degree 36, and some up to degree 50, and it also provided a very high order expansion for the tides.
- The JGM model (Joint Gravity Model) was produced jointly by NASA and the University of Texas. In 1994, JGM-2 (degree 70) amended GEMT3 (degree 50), itself successor to GEM-T2, with the first results from TOPEX/Poseidon, and JGM-3 integrated the data from other satellites suchas LAGEOS-2.

主要引力场模型

- The EGM model (Earth Gravity Model) is the result of a collaboration between SFC-NASA, NIMA (National Imagery and Mapping Agency),and OSU (Ohio State University), which has established many models, from OSU68 to OSU91. In 1996 came EGM96S of degree 70, with data provided solely by satellites, and EGM96 of degree 360, adjoining geophysical data.They used data from 40 satellites, including satellite to satellite measurements,with the GPS constellations26 and TDRSS. The latest model EGM2008 was based mainly on GRACE data to achieve an expansion up to degree 2190.
- Since 2002, the GGM model (GRACE Gravity Model) has been developed by the University of Texas using only data from GRACE: accelerometer, attitude, and distance between the two satellites (K-band range-rate).
- in Europe, the GRGS (Groupe de Recherche en G´eod´esie Spatiale) in France and the DGFI (Deutsches Geod¨atisches orschungsinstitut) in Germany have worked jointly to produce the GRIM model (GR for GRGS and IM for the institute in Munich) .

卫星大地测量估计地球引力常数

Method	Year	μ (km^3s^{-2})	Error
Lunar orbit	1959	398,620.	±6.
Explorer-27	1965	398,602.	±4.
Ranger-6, 7, 8, 9	1966	398,601.0	±0.7
Mariner-9	1971	398,601.2	±2.5
Venera-8	1972	398,600.4	±1.0
ATS-6 / GEOS-3	1979	398,600.40	±0.1
Laser/Moon	1985	398,600.444	±0.010
Laser/LAGEOS	1992	398,600.4415	±0.0008
Laser/LAGEOS	2000	398,600.4415	±0.0002


Q&A!