



中国科学院上海天文台

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# 变分方程

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# 状态量关于初值偏导数

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} \dot{\mathbf{r}}(t) \\ \ddot{\mathbf{r}}(t) \end{pmatrix} \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{p}) \\ \mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\tau, \mathbf{x}(\tau), \mathbf{p}) d\tau \end{cases}$$

$$\begin{cases} \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(t) = \mathbf{I} + \int_0^t \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\tau) \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\tau) d\tau \\ \Phi(t) = \Phi(0) + \int_0^t \mathbf{F}(\tau) \Phi(\tau) d\tau \\ \dot{\Phi}(t) = \mathbf{F}(t) \Phi(t), \quad \Phi(0) = \mathbf{I} \end{cases}$$

$$\Phi(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(t) \text{ and } \mathbf{F}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t).$$

# 状态量关于参数偏导数

$$\begin{cases} \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(t) = \int_0^t \left[ \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\tau) \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\tau) + \frac{\partial \mathbf{f}}{\partial \mathbf{p}}(\tau) \right] d\tau \\ \Psi(t) = \int_0^t [\mathbf{F}(\tau) \Psi(\tau) + \mathbf{G}(\tau)] d\tau \\ \dot{\Psi}(t) = \mathbf{F}(t) \Psi(t) + \mathbf{G}(t), \quad \Psi(0) = \mathbf{0} \end{cases}$$

$$\Psi(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(t) \text{ and } \mathbf{G}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{p}}(t).$$

# 二体的F矩阵

$$\ddot{\mathbf{r}} = -\frac{GM_{\oplus}}{r^3}\mathbf{r}$$

$$\frac{\partial r^n}{\partial \mathbf{r}} = \frac{\partial (x^2 + y^2 + z^2)^{n/2}}{\partial \mathbf{r}} = n \cdot r^{n-2} \cdot \mathbf{r}^T$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} = \frac{GM_{\oplus}}{r^5} \begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix}$$

# 非球形引力场F矩阵

$$U = \frac{GM}{r} \left[ 1 + \sum_{n=2}^{nmax} \sum_{m=0}^n \left[ \frac{a_e}{r} \right]^n P_m^n(\sin \phi') [C_{nm} \cos m\lambda + S_{nm} \sin m\lambda] \right]$$

变分方程需要计算矩阵  $[U_{2c}]_{i,j} = \frac{\partial^2 U}{\partial r_i \partial r_j}$

$$U_{2c} = C_1^T U_2 C_1 + \sum_{k=1}^3 \frac{\partial U}{\partial e_k} C_{2k}$$

$e_k$  ranges over the elements  $r$

$U_2$  is the matrix whose  $i, j^{th}$  element is given by  $\partial^2 U \partial e_i \partial e_j$

$C_1$  is the matrix whose  $i, j^{th}$  element is given by  $\frac{\partial e_i}{\partial r_j}$

$C_{2k}$  is the matrix whose  $i, j^{th}$  element is given by  $\frac{\partial^2 e_k}{\partial r_i \partial r_j}$

# 非球形引力场F矩阵

$$\frac{\partial^2 U}{\partial r^2} = \frac{2GM}{r^3} + \frac{GM}{r^3} \sum_{n=2}^{nmax} (n+1)(n+2) \left[ \frac{a_e}{r} \right]^n \\ \times \sum_{m=0}^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_n^m(\sin \phi')$$

$$\frac{\partial^2 U}{\partial r \partial \phi'} = \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left[ \frac{a_e}{r} \right]^n \sum_{m=0}^n (C_{nm} \cos m\lambda \\ + S_{nm} \sin m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin \phi')$$

$$\frac{\partial^2 U}{\partial r \partial \lambda} = \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left[ \frac{a_e}{r} \right]^n \sum_{m=0}^n m (-C_{nm} \sin m\lambda \\ + S_{nm} \cos m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin \phi')$$

# 非球形引力场F矩阵

$$\frac{\partial^2 U}{\partial \phi'^2} = \frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[ \frac{a_e}{r} \right]^n \sum_{m=0}^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \frac{\partial^2}{\partial \phi'^2} [P_n^m(\sin \phi')]$$

$$\frac{\partial^2 U}{\partial \phi' \partial \lambda} = \frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[ \frac{a_e}{r} \right]^n \sum_{m=0}^n m (-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin \phi')$$

$$\frac{\partial^2 U}{\partial \lambda^2} = -\frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[ \frac{a_e}{r} \right]^n \sum_{m=0}^n m^2 (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_n^m(\sin \phi')$$

# 非球形引力场F矩阵

$$\frac{\partial}{\partial \phi'} [P_n^m(\sin \phi')] = P_n^{m+1}(\sin \phi') - m \tan \phi' P_n^m(\sin \phi')$$

$$\begin{aligned} \frac{\partial^2}{\partial \phi'^2} [P_n^m(\sin \phi')] &= P_n^{m+2}(\sin \phi') - (m+1) \tan \phi' P_n^{m+1}(\sin \phi') \\ &\quad - m \tan \phi' [P_n^{m+1}(\sin \phi') - m \tan \phi' P_n^m(\sin \phi')] \\ &\quad - m \sec^2 \phi' P_n^m(\sin \phi') \end{aligned}$$

$$\frac{\partial U}{\partial \sin \phi'} = \frac{\partial U}{\partial \phi'} \frac{\partial \phi'}{\partial \sin \phi'}$$

$$\frac{\partial^2 U}{\partial \sin \phi'^2} = \frac{\partial \phi'}{\partial \sin \phi'} \left[ \frac{\partial^2 U}{\partial \phi'^2} \right] \frac{\partial \phi'}{\partial \sin \phi'} + \frac{\partial U}{\partial \phi'} \frac{\partial^2 \phi'}{\partial \sin \phi'^2}$$

$$\frac{\partial \phi'}{\partial \sin \phi} = \sec \phi'$$

$$\frac{\partial^2 \phi'}{\partial \sin \phi'^2} = \sin \phi' \sec^3 \phi'$$



# 非球形引力场F矩阵

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r}$$

$$\frac{\partial \sin \phi'}{\partial r_i} = -\frac{z r_i}{r^3} + \frac{1}{r} \frac{\partial x}{\partial r_i}$$

$$\frac{\partial \lambda}{\partial r_i} = \frac{1}{x^2 + y^2} \left[ x \frac{\partial y}{\partial r_i} - y \frac{\partial x}{\partial r_i} \right]$$

$$\frac{\partial^2 r}{\partial r_i \partial r_j} = \frac{r_i r_j}{r^3} + \frac{1}{r} \frac{\partial r_i}{\partial r_j}$$

$$\frac{\partial^2 \sin \phi'}{\partial r_i \partial r_j} = \frac{3z r_i r_j}{r^5} - \frac{1}{r^3} \left[ r_j \frac{\partial z}{\partial r_i} + r_i \frac{\partial z}{\partial r_j} + z \frac{\partial r_i}{\partial r_j} \right]$$

$$\frac{\partial^2 \lambda}{\partial r_i \partial r_j} = \frac{-2r_j}{(x^2 + y^2)^2} \left[ x \frac{\partial y}{\partial r_i} - y \frac{\partial x}{\partial r_i} \right] + \frac{1}{(x^2 + y^2)^2} \left[ \frac{\partial x}{\partial r_j} \frac{\partial y}{\partial r_j} - \frac{\partial y}{\partial r_j} \frac{\partial x}{\partial r_j} \right]$$

# 关于引力场系数的偏导数

$$\frac{\partial}{\partial C_{nm}} \left[ -\frac{\partial U}{\partial r} \right] = (n+1) \frac{GM}{r^2} \left[ \frac{a_e}{r} \right]^n \cos(m\lambda) P_n^m(\sin \phi')$$

$$\frac{\partial}{\partial C_{nm}} \left[ -\frac{\partial U}{\partial \lambda} \right] = m \frac{GM}{r} \left[ \frac{a_e}{r} \right]^n \sin(m\lambda) P_n^m(\sin \phi')$$

$$\frac{\partial}{\partial C_{nm}} \left[ -\frac{\partial U}{\partial \phi'} \right] = \frac{-GM}{r} \left[ \frac{a_e}{r} \right]^n \cos(m\lambda) \left[ P_n^{m+1}(\sin \phi') - m \tan \phi' P_n^m(\sin \phi') \right]$$

$$\frac{\partial}{\partial C_{nm}} \left[ -\frac{\partial U}{\partial x} \right] = \frac{\partial}{\partial C_{nm}} \left[ -\frac{\partial U}{\partial r} \right] \frac{\partial r}{\partial x} + \frac{\partial}{\partial C_{nm}} \left[ -\frac{\partial U}{\partial \lambda} \right] \frac{\partial \lambda}{\partial x} + \frac{\partial}{\partial C_{nm}} \left[ -\frac{\partial U}{\partial \phi'} \right] \frac{\partial \phi'}{\partial x}$$

# 第三体引力位

$$R_d = \frac{Gm_d}{r_d} \left[ \left[ 1 - \frac{2r}{r_d} S + \frac{r^2}{r_d^2} \right]^{-\frac{1}{2}} - \frac{r}{r_d} S \right]$$

$m_d$  is the mass of the disturbing body.

$\bar{r}_d$  is the geocentric true of date position vector to the disturbing body.

$S$  is equal to the cosine of the enclosed angle between  $\bar{r}$  and  $\bar{r}_d$ .

$\bar{r}$  is the geocentric true of date position vector of the satellite.

$G$  is the universal gravitational constant.

# 第三体A矩阵

$$\bar{a}_d = -Gm_d \left[ \frac{\bar{d}}{D} + \left[ \frac{\bar{r}}{r_d} \right] \right]$$

$$\bar{d} = \bar{r} - \bar{r}_d$$

$$D_d = [r_d^2 - 2rr_dS + r^2]^{\frac{3}{2}}$$

$$\frac{\partial^2 R_d}{\partial r_i \partial r_j} = -\frac{GMm_d}{D_d} \left[ \frac{\partial r_i}{\partial r_j} + \frac{3d_i d_j}{D_d^{\frac{2}{3}}} \right]$$

# 光压F矩阵

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} = +P_{\odot} C_r \frac{A}{m} \text{AU}^2 \left( \frac{1}{|\mathbf{r} - \mathbf{s}|^3} \mathbf{1}_{3 \times 3} - 3(\mathbf{r} - \mathbf{s}) \frac{(\mathbf{r} - \mathbf{s})^T}{|\mathbf{r} - \mathbf{s}|^5} \right)$$

$$\frac{\partial \ddot{\mathbf{r}}}{C_r} = \frac{1}{C_r} \ddot{\mathbf{r}} = -P_{\odot} \frac{A}{m} \frac{r_{\odot}}{r_{\odot}^3} \text{AU}^2$$

# 大气阻力F矩阵

$$\ddot{\mathbf{r}} = -\frac{1}{2} C_D \frac{A}{m} \rho \mathbf{v}_r \mathbf{v}_r \quad \text{with} \quad \mathbf{v}_r = \mathbf{v} - \boldsymbol{\omega}_{\oplus} \times \mathbf{r}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial C_D} = -\frac{1}{2} \frac{A}{m} \rho \mathbf{v}_r \mathbf{v}_r$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{v}} = -\frac{1}{2} C_D \frac{A}{m} \rho \left( \frac{\mathbf{v}_r \mathbf{v}_r^T}{v_r} + v_r \mathbf{1} \right)$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} = -\frac{1}{2} C_D \frac{A}{m} v_r \mathbf{v}_r \frac{\partial \rho}{\partial \mathbf{r}} - \frac{1}{2} C_D \frac{A}{m} \rho \left( \frac{\mathbf{v}_r \mathbf{v}_r^T}{v_r} + v_r \mathbf{1} \right) \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}$$

# 动力方程与变分方程联合积分（二阶）

$$\left\{ \begin{array}{l} \frac{d^2}{dt^2} \left( \frac{\partial \vec{R}}{\partial \vec{R}_0} \right) = \frac{\partial \vec{A}}{\partial \vec{R}} \frac{\partial \vec{R}}{\partial \vec{R}_0} + \frac{\partial \vec{A}}{\partial \dot{\vec{R}}} \frac{d}{dt} \left( \frac{\partial \vec{R}}{\partial \vec{R}_0} \right) \\ \frac{d^2}{dt^2} \left( \frac{\partial \vec{R}}{\partial \dot{\vec{R}}_0} \right) = \frac{\partial \vec{A}}{\partial \dot{\vec{R}}} \frac{\partial \vec{R}}{\partial \dot{\vec{R}}_0} + \frac{\partial \vec{A}}{\partial \ddot{\vec{R}}} \frac{d}{dt} \left( \frac{\partial \vec{R}}{\partial \dot{\vec{R}}_0} \right) \\ \frac{d^2}{dt^2} \left( \frac{\partial \vec{R}}{\partial \vec{p}_d} \right) = \frac{\partial \vec{A}}{\partial \vec{R}} \frac{\partial \vec{R}}{\partial \vec{p}_d} + \frac{\partial \vec{A}}{\partial \dot{\vec{R}}} \frac{d}{dt^2} \left( \frac{\partial \vec{R}}{\partial \vec{p}_d} \right) + \frac{\partial \vec{A}}{\partial \vec{p}_d} \end{array} \right.$$

$$\left( \frac{\partial \vec{R}}{\partial \vec{R}_0} \right)_{t=t_0} = I_{3 \times 3} \quad \left( \frac{\partial \vec{R}}{\partial \dot{\vec{R}}_0} \right)_{t=t_0} = 0, \quad \left( \frac{\partial \vec{R}}{\partial \vec{p}_d} \right)_{t=t_0} = 0$$

$$\left[ \frac{d}{dt} \left( \frac{\partial \vec{R}}{\partial \vec{R}_0} \right) \right]_{t=t_0} = 0, \quad \left[ \frac{d}{dt} \left( \frac{\partial \vec{R}}{\partial \vec{p}_d} \right) \right]_{t=t_0} = 0 \quad \left[ \frac{d}{dt} \left( \frac{\partial \vec{R}}{\partial \dot{\vec{R}}_0} \right) \right]_{t=t_0} = I_{3 \times 3}$$

# 动力方程与变分方程联合积分（二阶）

$$\vec{Z} = \begin{pmatrix} \bar{R} \\ \frac{\partial \bar{R}}{\partial x_0} \\ \frac{\partial \bar{R}}{\partial y_0} \\ \frac{\partial \bar{R}}{\partial z_0} \\ \frac{\partial \bar{R}}{\partial \dot{x}_0} \\ \frac{\partial \bar{R}}{\partial \dot{y}_0} \\ \frac{\partial \bar{R}}{\partial \dot{z}_0} \\ \frac{\partial \bar{R}}{\partial p_{d1}} \\ \vdots \\ \frac{\partial \bar{R}}{\partial p_{dNDP}} \end{pmatrix} \quad \frac{d^2 \vec{Z}}{dt^2} = \frac{d^2}{dt^2} \begin{pmatrix} \bar{R} \\ \frac{\partial \bar{R}}{\partial x_0} \\ \vdots \\ \frac{\partial \bar{R}}{\partial \dot{x}_0} \\ \vdots \\ \frac{\partial \bar{R}}{\partial p_{d1}} \\ \vdots \end{pmatrix} = \begin{pmatrix} \bar{A} \\ \frac{\partial \bar{A}}{\partial \bar{R}} \frac{\partial \bar{R}}{\partial x_0} + \frac{\partial \bar{A}}{\partial \dot{\bar{R}}} \frac{d}{dt} \left( \frac{\partial \bar{R}}{\partial x_0} \right) \\ \vdots \\ \frac{\partial \bar{A}}{\partial \bar{R}} \frac{\partial \bar{R}}{\partial \dot{x}_0} + \frac{\partial \bar{A}}{\partial \dot{\bar{R}}} \frac{d}{dt} \left( \frac{\partial \bar{R}}{\partial \dot{x}_0} \right) \\ \vdots \\ \frac{\partial \bar{A}}{\partial \bar{R}} \frac{\partial \bar{R}}{\partial p_{d1}} + \frac{\partial \bar{A}}{\partial \dot{\bar{R}}} \frac{d}{dt} \left( \frac{\partial \bar{R}}{\partial p_{d1}} \right) + \frac{\partial \bar{A}}{\partial p_{d1}} \\ \vdots \end{pmatrix} = \vec{f}$$



# 线性定常系统状态转移矩阵范例：

对于动力学系统

初始条件为

$$\dot{\mathbf{X}} = \begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(t_0) = x_{10}, x_2(t_0) = x_{20}$$

$$\Phi(t, t_0) \equiv \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)}$$



# 求解方法1

$$\frac{dx_1}{dt} = x_1$$

$$x_1(t) = ce^t, t = t_0, x_1 = x_{10}$$

$$c = x_{10}e^{-t_0}$$

$$x_1 = x_{10}e^{t-t_0}$$

$$dx_2 = \alpha x_{10}e^{t-t_0} dt$$

$$x_2 = \alpha x_{10}e^{t-t_0} + c_2, t = t_0, x_2 = x_{20}$$

$$x_2(t) = b + \alpha x_{10}(e^{t-t_0} - 1)$$

$$\Phi(t, t_0) \equiv \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)} = \begin{bmatrix} \frac{\partial x_1(t)}{\partial x_1(t_0)} & \frac{\partial x_1(t)}{\partial x_2(t_0)} \\ \frac{\partial x_2(t)}{\partial x_1(t_0)} & \frac{\partial x_2(t)}{\partial x_2(t_0)} \end{bmatrix} = \begin{bmatrix} e^{t-t_0} & 0 \\ \alpha(e^{t-t_0} - 1) & 1 \end{bmatrix}$$

## 求解方法2:

$$\begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad \Phi(t_0, t_0) = I$$

$$\begin{cases} \dot{\phi}_{11} = \phi_{11} & \phi_{11} = ce^t, \phi_{11}(t_0) = 1, c = e^{-t_0}, \phi_{11} = e^{(t-t_0)} \\ \dot{\phi}_{21} = \alpha\phi_{11} & \dot{\phi}_{21} = \alpha e^{t-t_0}, \phi_{21} = \alpha e^{t-t_0} + c, \phi_{21}(t_0) = 0 \\ \dot{\phi}_{12} = \phi_{12} & c = -\alpha & \phi_{21} = \alpha(e^{t-t_0} - 1) \\ \dot{\phi}_{22} = \alpha\phi_{12} & \phi_{12} = 0 & \phi_{22} = 1 \end{cases}$$



# Q&A!