



中国科学院上海天文台

Shanghai Astronomical Observatory, Chinese Academy of Science



中国科学院大学

University of Chinese Academy of Sciences

天体力学基础 二体问题

宋叶志
2020年秋季

作业邮箱: song.yz@foxmail.com

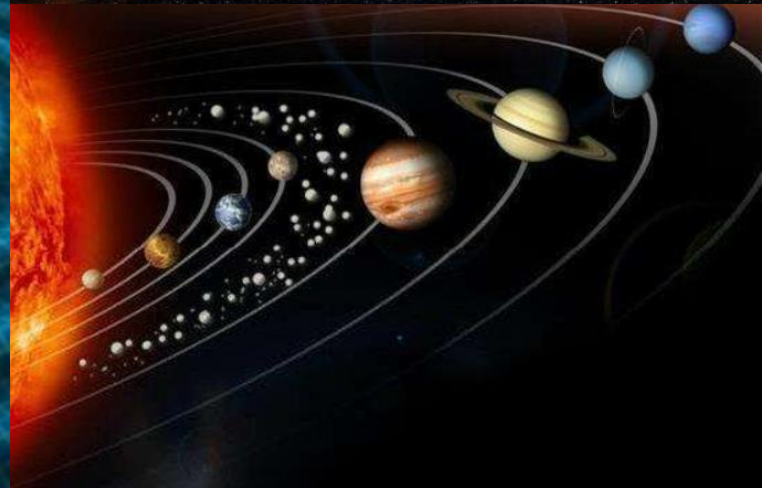
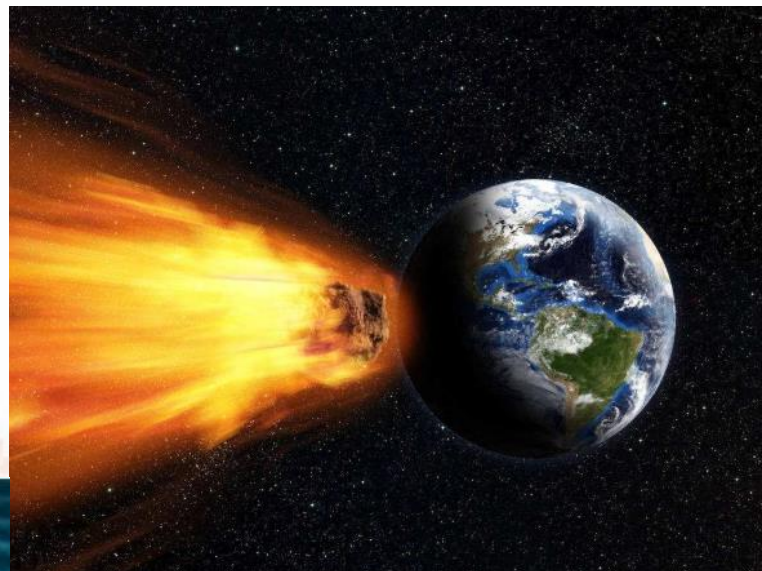
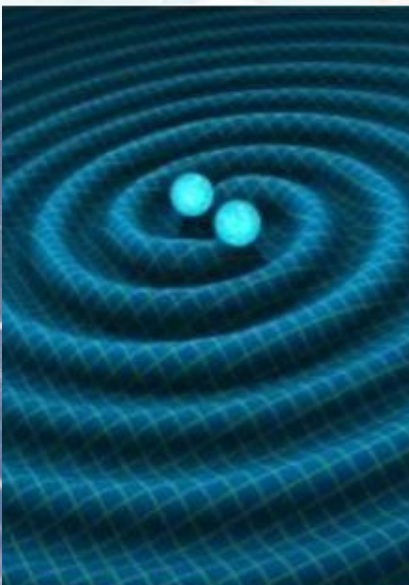
课件地址: <http://202.127.29.4/astrodynamics/course.php>

主要内容

- 天体力学概要
- 首次积分
- 椭圆运动展开式
- 无奇点根数、两行根数与正则根数描述
- 轨道空间几何

天体力学的研究对象

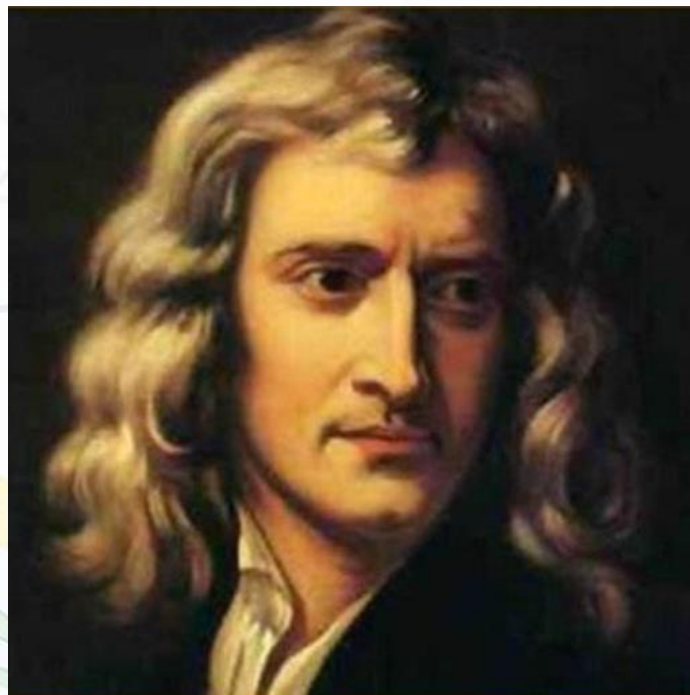
- ▶ 太阳系自然天体
- ▶ 地球卫星、深空探测器轨道力学
- ▶ 星系、星系团
- ▶ 太阳系外恒星-行星系统
- ▶ 宇宙学中的动力学问题
- ▶ 引力论与后牛顿天体力学
- ▶ 微分动力系统



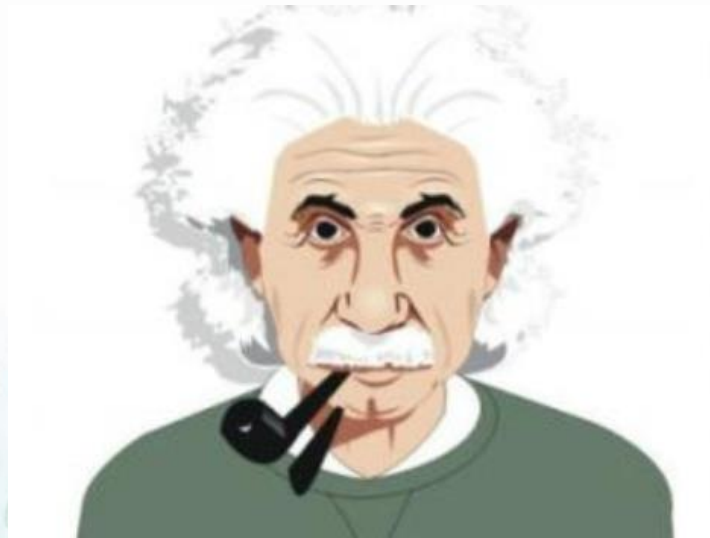
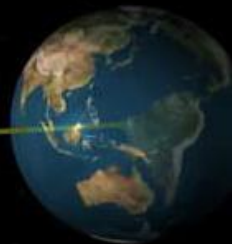
天体力学的发展



万有引力与参数化后牛顿引力方程



$$F = \frac{G \cdot m_1 \cdot m_2}{R^2}$$

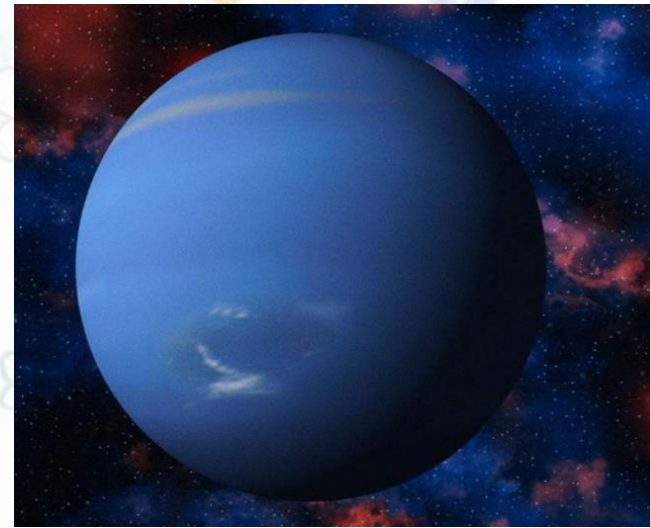


$$\begin{aligned} \ddot{\mathbf{x}}_i = & -k^2 \sum_{j=0, j \neq i}^n m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \left\{ 1 - \frac{2k^2(\beta + \gamma)}{c^2} \sum_{k=0, k \neq i}^n \frac{m_k}{|\mathbf{x}_i - \mathbf{x}_k|} \right. \\ & - \frac{k^2(2\beta - 1)}{c^2} \sum_{k=0, k \neq j}^n \frac{m_k}{|\mathbf{x}_j - \mathbf{x}_k|} + \gamma \frac{\dot{\mathbf{x}}_i^2}{c^2} + (1 + \gamma) \frac{\dot{\mathbf{x}}_j^2}{c^2} \\ & \left. - \frac{2(1 + \gamma)}{c^2} \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_j - \frac{3}{2c^2} \left[\frac{(\mathbf{x}_i - \mathbf{x}_j) \cdot \dot{\mathbf{x}}_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right]^2 - \frac{1}{2c^2} (\mathbf{x}_i - \mathbf{x}_j) \cdot \ddot{\mathbf{x}}_j \right\} \\ & + \frac{k^2}{c^2} \sum_{j=0, j \neq i}^n \frac{m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \left\{ (\mathbf{x}_i - \mathbf{x}_j) [(2 + 2\gamma) \dot{\mathbf{x}}_i \right. \\ & \left. - (1 + 2\gamma) \dot{\mathbf{x}}_j] \right\} \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) + \frac{k^2(3 + 4\gamma)}{2c^2} \sum_{j=0, j \neq i}^n m_j \frac{\ddot{\mathbf{x}}_j}{|\mathbf{x}_i - \mathbf{x}_j|}, \end{aligned}$$

天体力学的重大胜利： 哈雷彗星的回归与海王星的发现

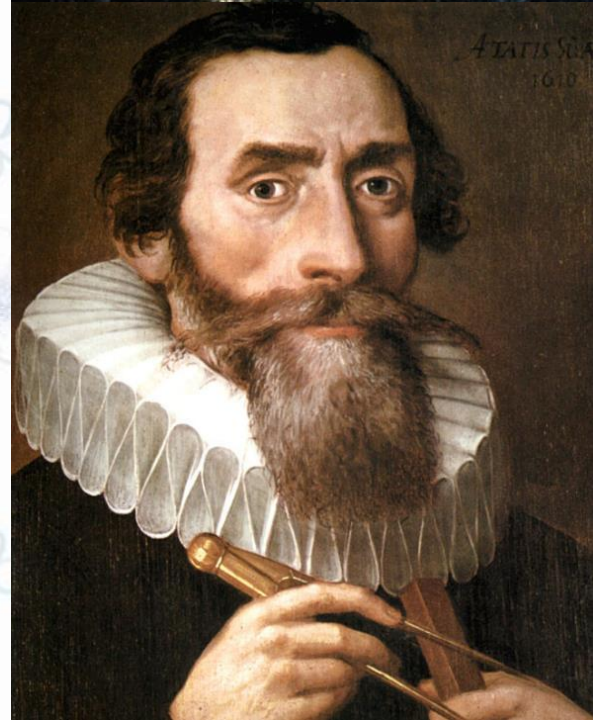
哈雷在1705年发表了《彗星天文学论说》，宣布1682年曾引起世人极大恐慌的大彗星，将于1758年再次出现于天空（后来他估计到木星可能影响到它的运动时，把回归的日期推迟到1759年）。当时哈雷已年过五十，知道在有生之年无缘再见到这颗大彗星了。哈雷去世10多年后，1758年底，这颗第一个被预报回归的彗星被一位业余天文学家观测到了。公元前240年起的每次回归我国都有所记载，最早的一次可能是周武王伐纣之年，即公元前1057年。哈雷彗星每隔大约76年都会按时回归。最近一次是1986年。

在1821年，Alexis Bouvard出版了天王星的轨道表，随后的观测显示出与表中的位置有越来越大的偏差，使得布瓦尔假设有一个摄动体存在。勒维耶完成了海王星位置的推算。在1846年9月23日晚间，海王星被发现了，与勒维耶预测的位置相距不到 1° 。



开普勒行星三定律

- 第一定律：各行星的轨道均为椭圆，太阳位于该椭圆的一个焦点上。
- 第二定律：行星与太阳的连线在相等时间内扫过的面积相等。
- 第三定律：行星轨道周期的平方与行星至太阳平均距离三次方成正比。



主要内容

- 天体力学概要
- 首次积分
- 椭圆运动展开式
- 无奇点根数、两行根数与正则根数描述
- 轨道空间几何

二体方程

定义中心天体势

$$\phi = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$

$$F(\mathbf{r}) = G \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}') d^3 \mathbf{r}' = -\nabla \phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

$$\nabla \cdot F(\mathbf{r}) = -G \rho(\mathbf{r}) \int_{|\mathbf{r} - \mathbf{r}'|=h} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^2 S'$$

$$\nabla^2 \phi = 4\pi G \rho$$

牛顿第一定理：球壳对位于其内部任意一点上的物体的引力之和为零。

牛顿第二定理：闭合球壳对位于球壳外任一物体的引力，等于把球壳所有质量集中于球壳中心上的点质量对该物体的引力。

因此把近球形（密度较为规则）大天体当作质点是一个很好的近似。

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3} \mathbf{r}$$

动量矩积分

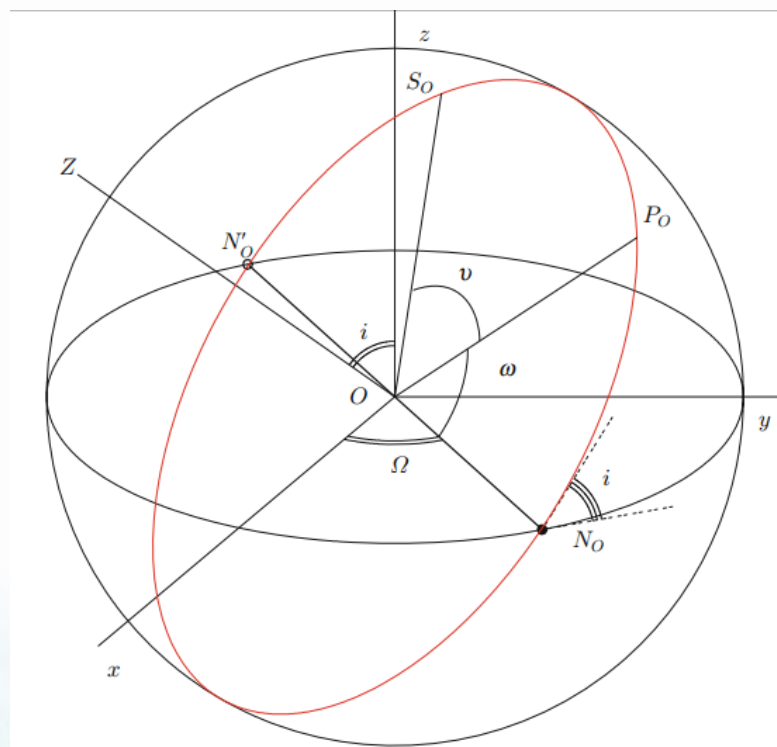
$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3} \mathbf{r}$$

$$\mathbf{r} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3} \mathbf{r} \times \mathbf{r} = 0$$

$$\frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} = 0$$

$$\mathbf{r} \times \dot{\mathbf{r}} = \text{constant}$$

$$\mathbf{R} = \frac{\mathbf{r} \times \mathbf{v}}{h} = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = \begin{pmatrix} \sin i \sin \Omega \\ -\sin i \cos \Omega \\ \cos i \end{pmatrix}$$



令该常数向量为 \mathbf{h} ，即为卫星的动量矩。由于 \mathbf{r} 和 $\dot{\mathbf{r}}$ 始终垂直于 \mathbf{h} 。因此其运动始终位于一个平面内，即轨道平面。动量矩 \mathbf{h} 的方向与卫星轨道面法线是平行的，称 \mathbf{h} 和坐标轴 Z 轴的夹角为轨道倾角 i 。轨道平面和赤道面的交线为节线 ON，节点为轨道升交点，节线 ON 与 X 轴的夹角称为升交点赤经 Ω 。轨道面倾角和升交点赤经决定了轨道运动平面的空间定向。+

正交曲线坐标表示的运动

笛卡尔坐标和正交曲线坐标之间的关系（其中， x, y, z 分别是 q_1, q_2, q_3 二阶连续可微函数）

$$\mathbf{r} = \mathbf{r}(q_1, q_2, q_3) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{e}_i = \frac{1}{H_i} \frac{\partial \mathbf{r}}{\partial q_i}, \quad \frac{\partial \mathbf{r}}{\partial q_i} = \frac{\partial x}{\partial q_i} \mathbf{i} + \frac{\partial y}{\partial q_i} \mathbf{j} + \frac{\partial z}{\partial q_i} \mathbf{k}$$

$$H_i = \left| \frac{\partial \mathbf{r}}{\partial q_i} \right| = \sqrt{\left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2}$$

拉梅系数

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{r}}{\partial q_2} \dot{q}_2 + \frac{\partial \mathbf{r}}{\partial q_3} \dot{q}_3 = v_{q_1} \mathbf{e}_1 + v_{q_2} \mathbf{e}_2 + v_{q_3} \mathbf{e}_3$$

$$v_{q_i} = H_i \dot{q}_i$$

正交曲线坐标系加速度表达式

$$w_{q_i} = \frac{d\mathbf{v}}{dt} \cdot \mathbf{e}_i = \frac{1}{H_i} \left(\frac{d\mathbf{v}}{dt} \cdot \frac{\partial \mathbf{r}}{\partial q_i} \right) = \frac{1}{H_i} \left[\frac{d}{dt} \left(\mathbf{v} \cdot \frac{\partial \mathbf{r}}{\partial q_i} \right) - \mathbf{v} \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}}{\partial q_i} \right) \right]$$

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}}{\partial q_i} \right) = \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_1} \dot{q}_1 + \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_2} \dot{q}_2 + \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_3} \dot{q}_3$$

$$\frac{\partial \mathbf{v}}{\partial q_i} = \frac{\partial^2 \mathbf{r}}{\partial q_1 \partial q_i} \dot{q}_1 + \frac{\partial^2 \mathbf{r}}{\partial q_2 \partial q_i} \dot{q}_2 + \frac{\partial^2 \mathbf{r}}{\partial q_3 \partial q_i} \dot{q}_3$$

$$w_{q_i} = \frac{1}{H_i} \left[\frac{d}{dt} \left(\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \dot{q}_i} \right) - \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_i} \right] \quad \text{代入 } T = v^2/2$$

$$w_{q_i} = \frac{1}{H_i} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right)$$

柱坐标与球坐标速度与加速度表达式

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z; \quad H_r = 1, \quad H_\varphi = r, \quad H_z = 1$$

$$v_r = \dot{r}, \quad v_\varphi = r\dot{\varphi}, \quad v_z = \dot{z};$$

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2);$$

$$w_r = \ddot{r} - r\dot{\varphi}^2, \quad w_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi}, \quad w_z = \ddot{z}.$$

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta; \quad H_r = 1, \quad H_\varphi = r \sin \theta, \quad H_\theta = r;$$

$$v_r = \dot{r}, \quad v_\varphi = r \sin \theta \dot{\varphi}, \quad v_\theta = r\dot{\theta};$$

$$T = \frac{1}{2}(\dot{r}^2 + r^2 \sin^2 \theta \dot{\varphi}^2 + r^2 \dot{\theta}^2);$$

$$w_r = \ddot{r} - r \sin^2 \theta \dot{\varphi}^2 - r\dot{\theta}^2, \quad w_\varphi = r \sin \theta \ddot{\varphi} + 2 \sin \theta \dot{r} \dot{\varphi} + 2r \cos \theta \dot{\varphi} \dot{\theta},$$

$$w_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r \sin \theta \cos \theta \dot{\varphi}^2.$$

平面内运动（极坐标）

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases} \quad \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = 0 \quad r^2 \dot{\theta} = \text{constant}$$

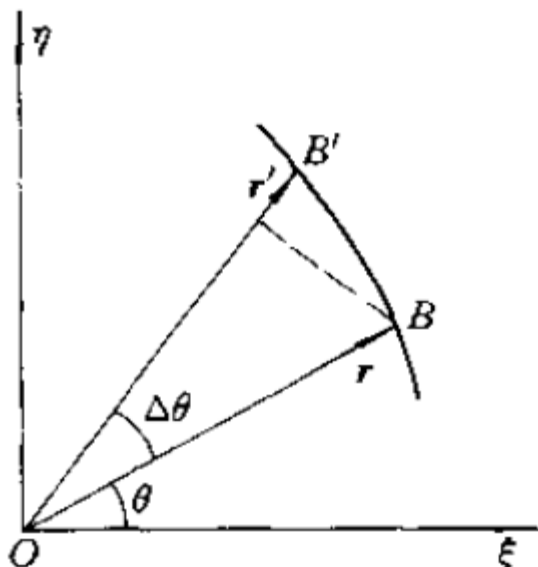
用 $u=1/r$ 代替 r , 可以求出 r 关于 θ 的导函数

$$\begin{cases} \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \frac{d}{d\theta} \left(\frac{1}{u} \right) \frac{d\theta}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} \dot{\theta} = -h \frac{du}{d\theta} \\ \ddot{r} = \frac{d\dot{r}}{dt} = \frac{d}{dt} \left(-h \frac{du}{d\theta} \right) = \frac{d}{d\theta} \left(-h \frac{du}{d\theta} \right) \dot{\theta} = -h^2 u^2 \frac{d^2 u}{d\theta^2} \end{cases}$$

开普勒第一定律：椭圆方程

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2} \quad \frac{1}{r} = \frac{\mu}{h^2} [1 + e \cos(\theta - \omega)] \quad r = \frac{p}{1 + e \cos(\theta - \omega)}$$

平面内运动



$$\Delta A = \frac{1}{2} r r' \sin \Delta \theta$$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r r' \frac{\Delta \theta}{\Delta t} \frac{\sin \Delta \theta}{\Delta \theta}$$

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h$$

开普勒第二定律

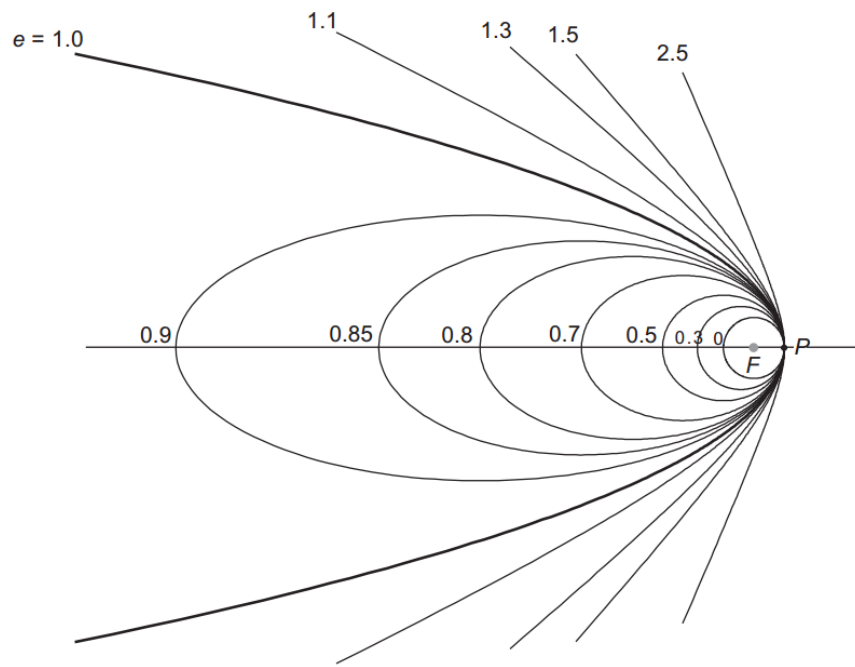
卫星一个周期 T 内，扫过的面积为 $\pi a b$ ，有 $\pi a b = \frac{1}{2} h T$

$$a = \frac{h^2}{\mu(1-e^2)}, a(1-e^2) = b\sqrt{1-e^2}, e = \sqrt{1 - \frac{b^2}{a^2}}$$

开普勒第三定律 $T = 2\pi \sqrt{\frac{a^3}{\mu}}$

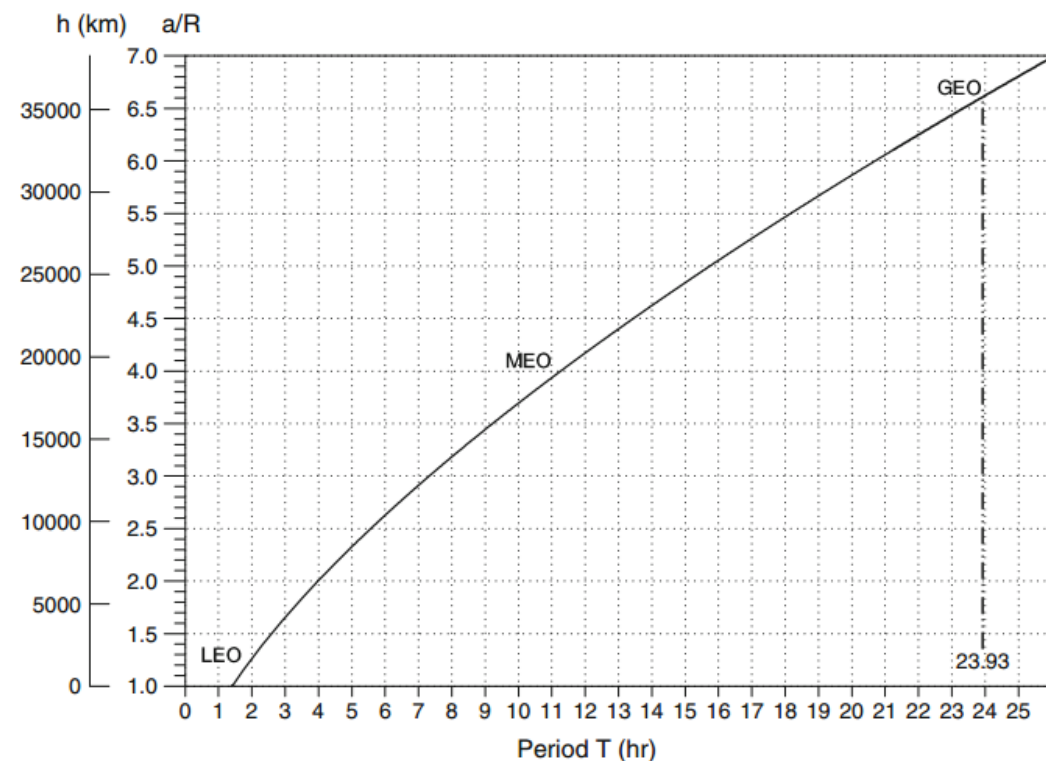
$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

抛物与双曲轨道



Type	Eccentricity	Semi-latus rectum	Perihelion	Energy
Circle	$e = 0$	$p = a$	a	$-\frac{\mu}{2a} < 0$
Ellipse	$e < 1$	$p = a(1 - e^2)$	$a(1 - e)$	$-\frac{\mu}{2a} < 0$
Parabola	$e = 1$	p	$q = \frac{p}{2}$	$= 0$
Hyperbola	$e > 1$	$p = a(e^2 - 1)$	$a(e - 1)$	$+\frac{\mu}{2a} > 0$

地球卫星近圆轨道高度与周期的关系



```
song@SONG-PC F:\test
```

```
$ satperiod 400 20
```

a(km)	Height(km)	T(minute)	T(hour)
6378.14	0.00	84.49	1.41
6778.14	400.00	92.56	1.54
7178.14	800.00	100.87	1.68
7578.14	1200.00	109.42	1.82
7978.14	1600.00	118.20	1.97
8378.14	2000.00	127.20	2.12
8778.14	2400.00	136.42	2.27
9178.14	2800.00	145.85	2.43
9578.14	3200.00	155.48	2.59
9978.14	3600.00	165.32	2.76
10378.14	4000.00	175.36	2.92
10778.14	4400.00	185.60	3.09
11178.14	4800.00	196.03	3.27
11578.14	5200.00	206.64	3.44
11978.14	5600.00	217.44	3.62
12378.14	6000.00	228.42	3.81
12778.14	6400.00	239.59	3.99
13178.14	6800.00	250.92	4.18
13578.14	7200.00	262.43	4.37
13978.14	7600.00	274.12	4.57
14378.14	8000.00	285.97	4.77

真近点角作为时间的函数

$$\frac{d\theta}{dt} = \frac{h}{r^2} \quad \frac{\mu^2}{h^3} dt = \frac{d\theta}{(1 + e \cos \theta)^2}$$

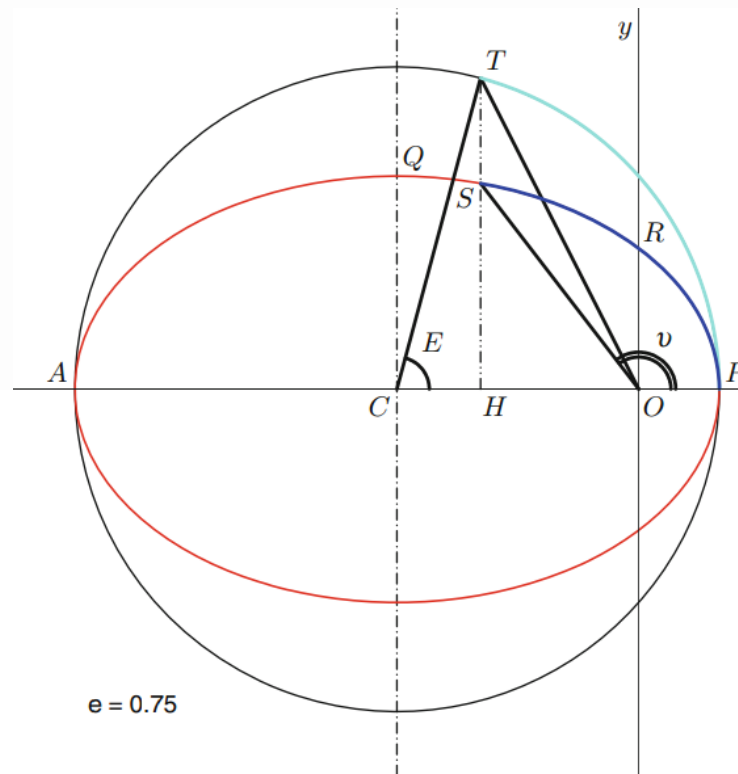
$$\frac{\mu^2}{h^3} (t - t_p) = \int_0^\theta \frac{d\vartheta}{(1 + e \cos \vartheta)^2}$$

Table 4.2. Equation for the true anomaly v : $\int_0^v \frac{dv'}{(1 + e \cos v')^2} = \sqrt{\frac{\mu}{p^3}} (t - T_0)$

Type	Transformation	Solution
Circle	—	$v = \sigma(t)$ *)
Ellipse	$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$	$E - e \sin E = \sigma(t)$ *)
Parabola	—	$\tan \frac{v}{2} + \frac{1}{3} \tan^3 \frac{v}{2} = 2 \sqrt{\frac{\mu}{p^3}} (t - T_0)$
Hyperbola	$\tan \frac{v}{2} = \sqrt{\frac{e+1}{e-1}} \tanh \frac{F}{2}$	$e \sinh F - F = \sigma(t)$ *)

*) where: $\sigma(t) \stackrel{\text{def}}{=} n (t - T_0)$, with: $n \stackrel{\text{def}}{=} \sqrt{\frac{\mu}{a^3}}$

$$n(t - t_p) = M = E - e \sin E$$



真近点角、偏近点角与平近点角

$$\sin f = \frac{(1 - e^2)^{1/2} \sin E}{1 - e \cos E}$$

$$\sin E = \frac{(1 - e^2)^{1/2} \sin f}{1 + e \cos f}$$

$$\cos f = \frac{\cos E - e}{1 - e \cos E}$$

$$\cos E = \frac{e + \cos f}{1 + e \cos f}$$

$$\tan \frac{f}{2} = \left(\frac{1 + e}{1 - e} \right)^{1/2} \tan \frac{E}{2}$$

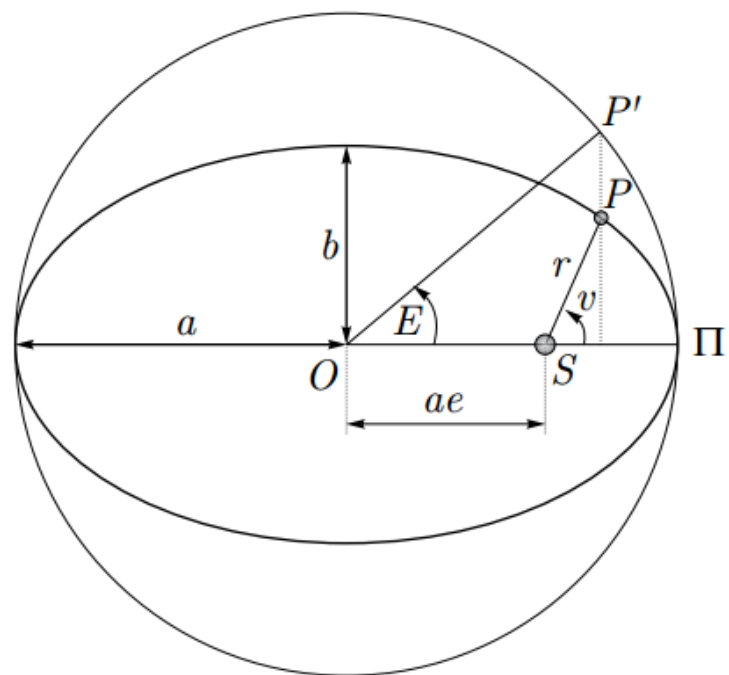
$$\tan \frac{E}{2} = \left(\frac{1 - e}{1 + e} \right)^{1/2} \tan \frac{f}{2}$$

$$M = E - e \sin E \quad (\text{Kepler's Equation})$$

$$\frac{df}{dE} = \frac{(1 - e^2)^{1/2}}{1 - e \cos E} = \frac{1 + e \cos f}{(1 - e^2)^{1/2}}$$

$$\frac{dM}{dE} = 1 - e \cos E = \frac{1 - e^2}{1 + e \cos f}$$

$$\frac{dM}{df} = \frac{(1 - e \cos E)^2}{(1 - e^2)^{1/2}} = \frac{(1 - e^2)^{3/2}}{(1 + e \cos f)^2}$$



开普勒方程数值解

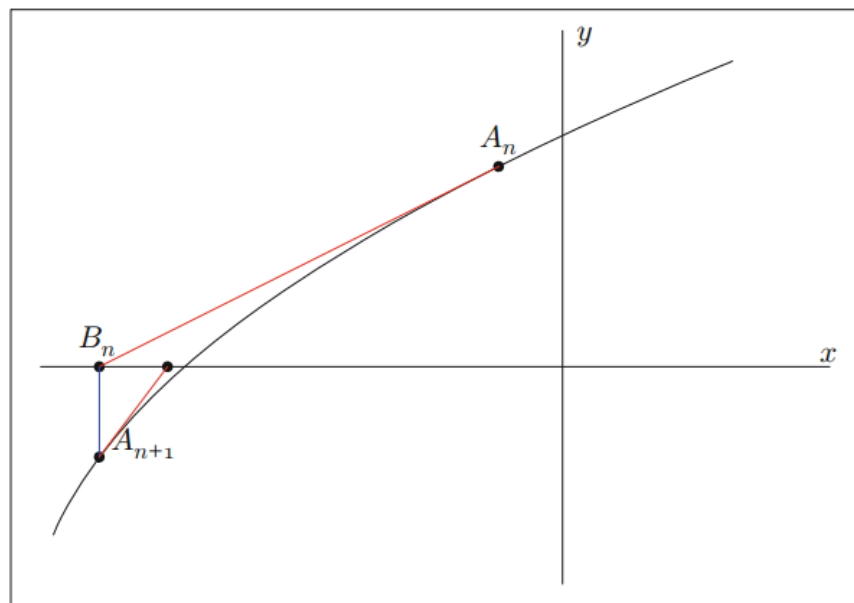
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(E) = E - e \sin E - M$$

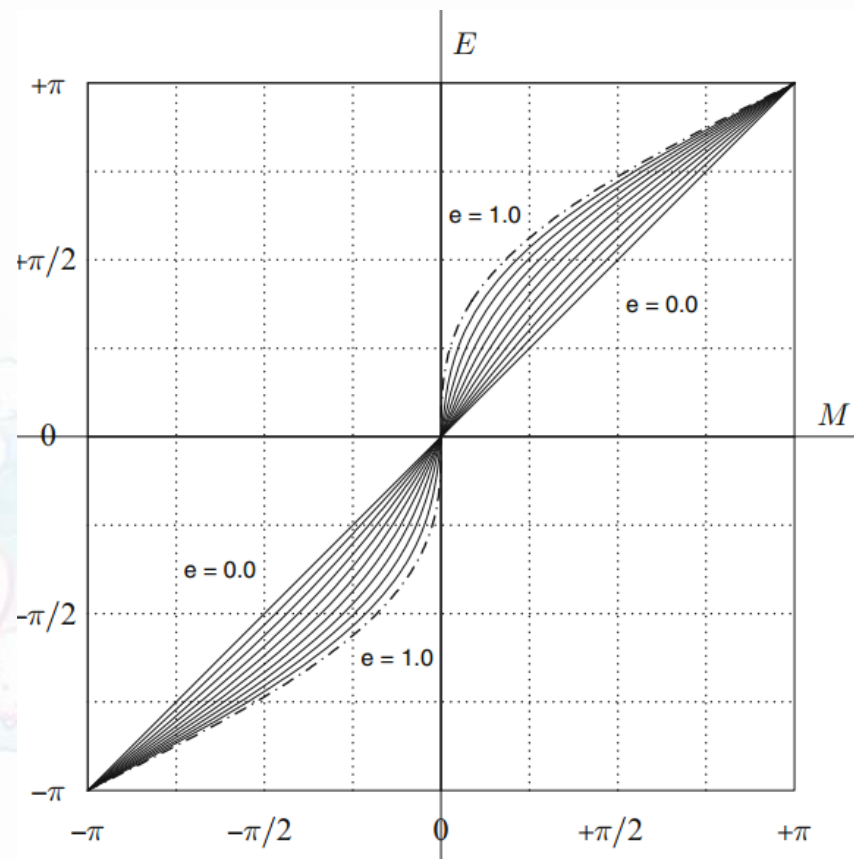
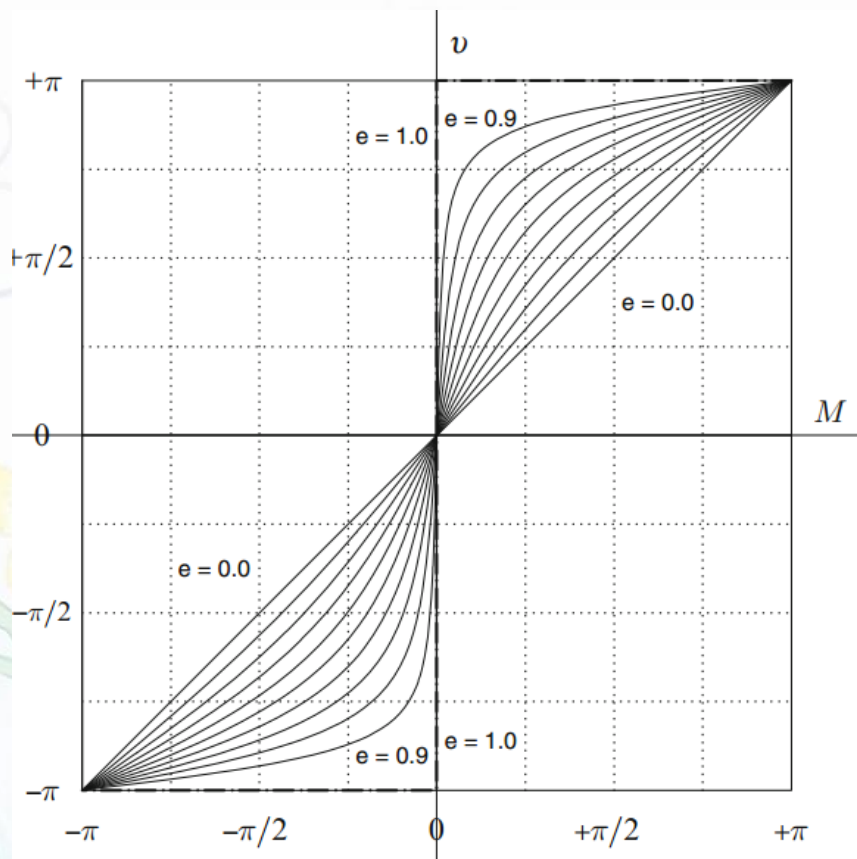
$$f'(E) = 1 - e \cos E$$

$$E_{n+1} = E_n - \frac{E_n - e \sin E_n - M}{1 - e \cos E_n}$$

$$v = 2 \arctan \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right)$$



近点角关系图



开普勒轨道根数

▶ 形状:

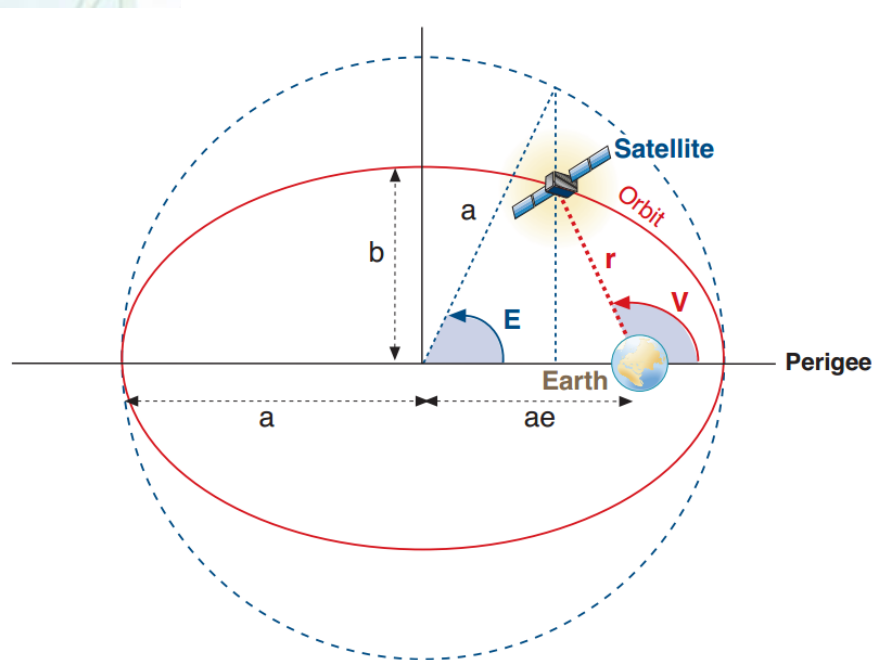
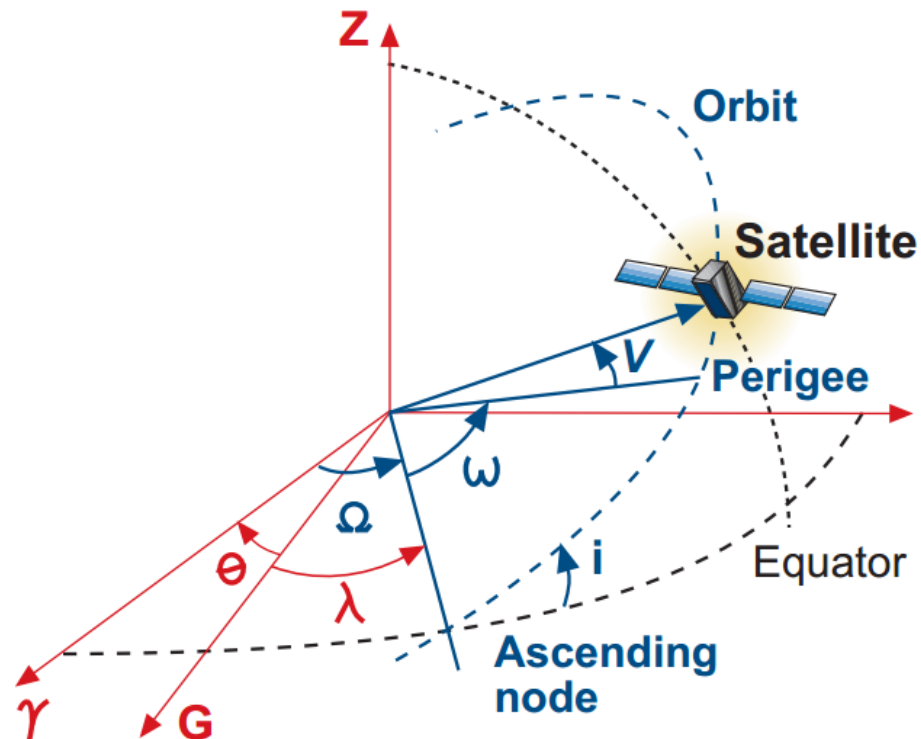
- a = 半长径
- e = 偏心率

▶ 空间定向:

- i = 轨道倾角
- Ω = 升交点赤经
- ω = 近星点幅角

▶ 位置:

- v = 真近点角



根数与位置速度

真近点角、平近点角、偏近点角度及过近星点时刻关系

$$M(t) = n(t - T_0)$$

$$E(t) = M(t) + e \sin E(t)$$

$$\tan \frac{v}{2} = \left(\frac{1+e}{1-e} \right)^{\frac{1}{2}} \tan \frac{E}{2}$$

$$n = \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}}$$

$$\begin{aligned} \vec{r} &= r\hat{r} = r \cos f \hat{P} + r \sin f \hat{Q} \\ &= a(\cos E - e) \hat{P} + a\sqrt{1-e^2} \sin E \hat{Q} \end{aligned}$$

$$\begin{aligned} \dot{\vec{r}} &= -\sqrt{\frac{\mu}{p}} \left[\sin f \hat{P} - (\cos f + e) \hat{Q} \right] \\ &= -\frac{\sqrt{\mu a}}{r} \left[\sin E \hat{P} - \sqrt{1-e^2} \cos E \hat{Q} \right] \end{aligned}$$

$$\hat{P} = \begin{pmatrix} \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i \\ \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i \\ \sin \omega \sin i \end{pmatrix}$$

$$\hat{Q} = \begin{pmatrix} -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i \\ \sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i \\ \cos \omega \sin i \end{pmatrix}$$

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}$$

$$\begin{cases} e \cos E = 1 - \frac{r}{a} \\ e \sin E = r\dot{r} / \sqrt{\mu a} \end{cases}$$

$$M = E - e \sin E$$

$$\begin{cases} P_z = \sin i \sin \omega, & Q_z = \sin i \cos \omega \\ \begin{pmatrix} R_x \\ -R_y \\ R_z \end{pmatrix} = \begin{pmatrix} \sin i \sin \Omega \\ \sin i \cos \Omega \\ \cos i \end{pmatrix} \end{cases}$$

$$\begin{cases} \omega = \tan^{-1}(P_z/Q_z) \\ \Omega = \tan^{-1}(R_x/(-R_y)) \\ i = \cos^{-1} R_z \end{cases}$$

开普勒根数与位置速度算例

16 Sep 2019 01:01:39

Satellite-JASON-2_33105: J2000 Position & Velocity

Time (UTCG)	x (km)	y (km)	z (km)	vx (km/sec)	vy (km/sec)	vz (km/sec)
16 Sep 2019 04:00:00.000	-5291.777394	-845.038485	-5558.116835	-3.472599	-4.820868	4.034093
16 Sep 2019 04:01:00.000	-5491.791680	-1132.825631	-5307.530043	-3.192813	-4.769552	4.316635
16 Sep 2019 04:02:00.000	-5674.714794	-1417.087258	-5040.388879	-2.903040	-4.703381	4.585762
16 Sep 2019 04:03:00.000	-5839.974295	-1696.937917	-4757.523749	-2.604178	-4.622557	4.840631
16 Sep 2019 04:04:00.000	-5987.052414	-1971.505544	-4459.814330	-2.297151	-4.527326	5.080443
16 Sep 2019 04:05:00.000	-6115.487742	-2239.934180	-4148.186898	-1.982912	-4.417977	5.304444
16 Sep 2019 04:06:00.000	-6224.876737	-2501.386655	-3823.611511	-1.662437	-4.294847	5.511930
16 Sep 2019 04:07:00.000	-6314.875062	-2755.047226	-3487.099033	-1.336723	-4.158314	5.702244

16 Sep 2019 01:03:43

Satellite-JASON-2_33105: J2000 Classical Orbit Elements

Time (UTCG)	Semi-major Axis (km)	Eccentricity	Inclination (deg)	RAAN (deg)	Arg of Perigee (deg)	True Anomaly (deg)	Mean Anomaly (deg)
16 Sep 2019 04:00:00.000	7712.709754	0.001157	65.972	216.614	153.922	154.061	154.003
16 Sep 2019 04:01:00.000	7713.491720	0.001079	65.974	216.612	157.619	153.563	153.508
16 Sep 2019 04:02:00.000	7714.285891	0.000997	65.975	216.611	161.251	153.129	153.077
16 Sep 2019 04:03:00.000	7715.082398	0.000911	65.976	216.609	164.781	152.797	152.749
16 Sep 2019 04:04:00.000	7715.871334	0.000822	65.978	216.608	168.157	152.619	152.576
16 Sep 2019 04:05:00.000	7716.642885	0.000731	65.979	216.607	171.296	152.678	152.640

主要内容

- 天体力学概要
- 首次积分
- 椭圆运动展开式
- 无奇点根数、两行根数与正则根数描述
- 轨道空间几何

第一类贝塞尔函数

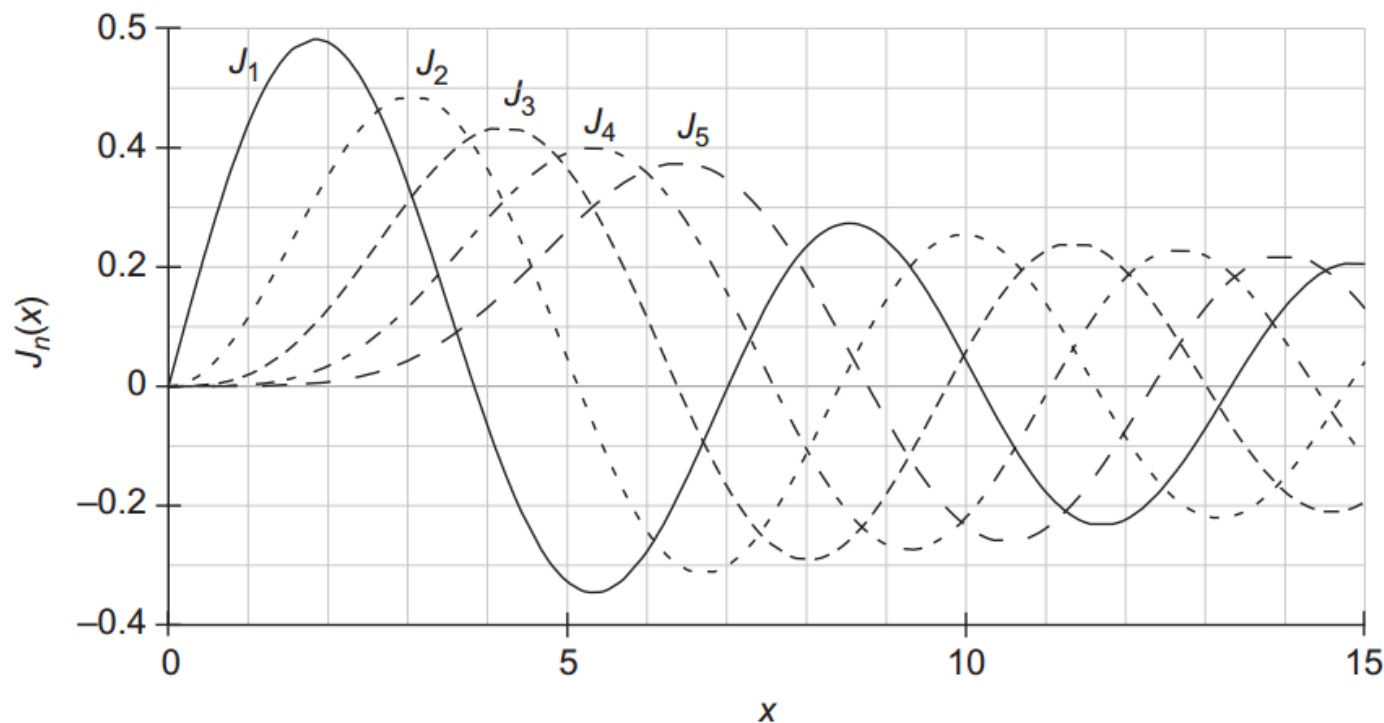
where $J_k(x)$ is the Bessel function

$$J_k(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(kt - x \sin t) dt$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

Using the property of the Bessel functions

$$J_k(x) = \frac{x}{2k} [J_{k-1}(x) + J_{k+1}(x)]$$



偏近点角的级数展开

$$E = M + \sum_{k=1}^{\infty} \frac{2J_k(ke)}{k} \sin kM.$$

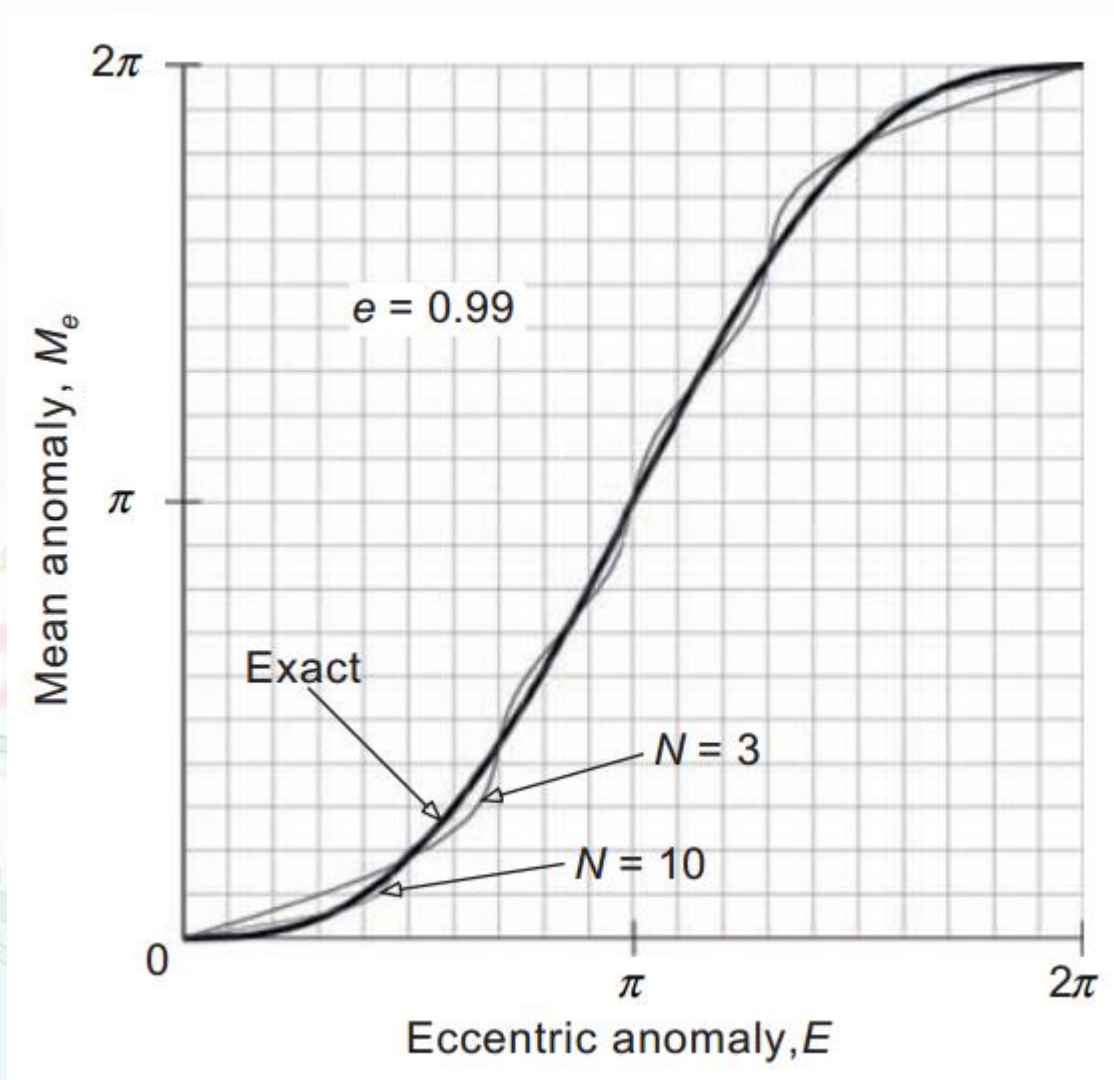
$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{4} \left(\frac{x}{2}\right)^4 - \cdots + \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} + \cdots,$$

$$J_k(x) = \left(\frac{x}{2}\right)^k \frac{1}{k!} \left[1 - \frac{1}{k+1} \left(\frac{x}{2}\right)^2 + \cdots + \frac{(-1)^n}{n!(k+1)(k+2)\cdots(k+n)} \left(\frac{x}{2}\right)^{2n} + \cdots \right].$$

Neglecting terms higher than e^3 we have the approximation

$$\begin{aligned} E &= M + \left(e - \frac{e^3}{8} \right) \sin M + \frac{e^2}{2} \sin 2M + \frac{3e^3}{8} \sin 3M \\ &= M + e \sin M + \frac{e^2}{2} \sin 2M + \frac{e^3}{8} (-\sin M + 3 \sin 3M). \end{aligned}$$

贝塞尔函数逼近性能



Sin(nE)与cos(nE)展开

$$\cos nE = -\frac{e}{2}\delta_{n1} + \sum_{k=1}^{\infty} \frac{n}{k} [J_{k-n}(ke) - J_{k+n}(ke)] \cos kM.$$

$$\sin nE = \sum_{k=1}^{\infty} \frac{n}{k} [J_{k-n}(ke) + J_{k+n}(ke)] \sin kM$$

$$\begin{aligned} \cos E &= -\frac{e}{2} + \left(1 - \frac{3e^2}{8}\right) \cos M + \left(\frac{e}{2} - \frac{e^3}{2}\right) \cos 2M \\ &\quad + \frac{3e^2}{8} \cos 3M + \frac{e^3}{3} \cos 4M, \end{aligned}$$

$$\begin{aligned} \cos 2E &= \left(-e + \frac{e^3}{12}\right) \cos M + (1 - e^2) \cos 2M \\ &\quad + \left(e - \frac{9e^3}{8}\right) \cos 3M + e^2 \cos 4M + \frac{25e^3}{24} \cos 5M, \end{aligned}$$

$$\begin{aligned} \sin E &= \left(1 - \frac{e^2}{8}\right) \sin M + \left(\frac{e}{2} - \frac{e^3}{8}\right) \sin 2M \\ &\quad + \frac{3e^2}{8} \sin 3M + \frac{e^3}{3} \sin 4M, \end{aligned}$$

$$\begin{aligned} \sin 2E &= \left(-e + \frac{e^3}{6}\right) \sin M + (1 - e^2) \sin 2M \\ &\quad + \left(e - \frac{9e^3}{8}\right) \sin 3M + e^2 \sin 4M + \frac{25e^3}{24} \sin 5M. \end{aligned}$$

距离作为平近点角的展开

$$\begin{aligned}\frac{r}{a} &= 1 - e \cos E \\ &= 1 + \frac{e^2}{2} - \sum_{k=1}^{\infty} \frac{e}{k} (J_{k-1}(ke) - J_{k+1}(ke)) \cos kM \\ &= 1 + \frac{e^2}{2} - \sum_{k=1}^{\infty} \frac{2e}{k^2} \frac{dJ_k(ke)}{de} \cos kM.\end{aligned}$$

Keeping only the terms up to e^3 we get

$$\frac{r}{a} = 1 + \frac{e^2}{2} + \left(-e + \frac{3e^3}{8}\right) \cos M - \frac{e^2}{2} \cos 2M - \frac{3e^3}{8} \cos 3M.$$

$$\begin{aligned}\frac{a}{r} &= \frac{1}{1 - e \cos E} = \frac{1}{dM/dE} = \frac{dE}{dM} \\ &= 1 + \sum_{k=1}^{\infty} 2J_k(ke) \cos kM.\end{aligned}$$

$$\frac{a}{r} = 1 + \left(e - \frac{e^3}{8}\right) \cos M + e^2 \cos 2M + \frac{9e^3}{8} \cos 3M.$$

超几何方程与超几何函数

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

方程系数的每个奇点都是方程的奇点。系数都是解析函数的其余一切值，称为方程的常点。方程三个奇点 $0, 1, \infty$

$$\begin{aligned} F(a, b, c; x) &= 1 + \frac{ab}{c} \frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \dots \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!}, \end{aligned}$$

$$y(x) = AF(a, b, c; x) + Bx^{1-c}F(a-c+1, b-c+1, 2-c; x)$$

超几何函数特例

α	β	γ	z	F
$-n$	β	γ	x	$\sum_{k=0}^n \frac{(-n)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!}$, where $n = 1, 2, \dots$
$-n$	β	$-n - m$	x	$\sum_{k=0}^n \frac{(-n)_k (\beta)_k}{(-n - m)_k} \frac{x^k}{k!}$, where $n = 1, 2, \dots$
α	β	β	x	$(1 - x)^{-\alpha}$
α	$\alpha + 1$	$\frac{1}{2}\alpha$	x	$(1 + x)(1 - x)^{-\alpha - 1}$
α	$\alpha + \frac{1}{2}$	$2\alpha + 1$	x	$\left(\frac{1 + \sqrt{1 - x}}{2}\right)^{-2\alpha}$
α	$\alpha + \frac{1}{2}$	2α	x	$\frac{1}{\sqrt{1 - x}} \left(\frac{1 + \sqrt{1 - x}}{2}\right)^{1 - 2\alpha}$
α	$\alpha + \frac{1}{2}$	$\frac{3}{2}$	x^2	$\frac{(1 + x)^{1 - 2\alpha} - (1 - x)^{1 - 2\alpha}}{2x(1 - 2\alpha)}$
α	$\alpha + \frac{1}{2}$	$\frac{1}{2}$	$-\tan^2 x$	$\cos^{2\alpha} x \cos(2\alpha x)$
α	$\alpha + \frac{1}{2}$	$\frac{1}{2}$	x^2	$\frac{1}{2} [(1 + x)^{-2\alpha} + (1 - x)^{-2\alpha}]$
α	$\frac{1}{2}$	$\alpha - 1$	x	$2^{2\alpha - 2} (1 - \sqrt{1 - x})^{2 - 2\alpha}$

勒让德方程解的超几何级数表示

$$(1 - z^2)y''_{zz} - 2zy'_z + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] y = 0.$$

For $|1 - z| < 2$, the formulas

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{z + 1}{z - 1} \right)^{\mu/2} F\left(-\nu, 1 + \nu, 1 - \mu; \frac{1 - z}{2}\right),$$
$$Q_\nu^\mu(z) = A \left(\frac{z - 1}{z + 1} \right)^{\frac{\mu}{2}} F\left(-\nu, 1 + \nu, 1 + \mu; \frac{1 - z}{2}\right) + B \left(\frac{z + 1}{z - 1} \right)^{\frac{\mu}{2}} F\left(-\nu, 1 + \nu, 1 - \mu; \frac{1 - z}{2}\right)$$
$$A = e^{i\mu\pi} \frac{\Gamma(-\mu)\Gamma(1 + \nu + \mu)}{2\Gamma(1 + \nu - \mu)}, \quad B = e^{i\mu\pi} \frac{\Gamma(\mu)}{2}, \quad i^2 = -1,$$

For $|z| > 1$,

$$P_\nu^\mu(z) = \frac{2^{-\nu-1}\Gamma(-\frac{1}{2} - \nu)}{\sqrt{\pi}\Gamma(-\nu - \mu)} z^{-\nu+\mu-1} (z^2 - 1)^{-\mu/2} F\left(\frac{1 + \nu - \mu}{2}, \frac{2 + \nu - \mu}{2}, \frac{2\nu + 3}{2}; \frac{1}{z^2}\right)$$
$$+ \frac{2^\nu\Gamma(\frac{1}{2} + \nu)}{\Gamma(1 + \nu - \mu)} z^{\nu+\mu} (z^2 - 1)^{-\mu/2} F\left(-\frac{\nu + \mu}{2}, \frac{1 - \nu - \mu}{2}, \frac{1 - 2\nu}{2}; \frac{1}{z^2}\right),$$
$$Q_\nu^\mu(z) = e^{i\pi\mu} \frac{\sqrt{\pi}\Gamma(\nu + \mu + 1)}{2^{\nu+1}\Gamma(\nu + \frac{3}{2})} z^{-\nu-\mu-1} (z^2 - 1)^{\mu/2} F\left(\frac{2 + \nu + \mu}{2}, \frac{1 + \nu + \mu}{2}, \frac{2\nu + 3}{2}; \frac{1}{z^2}\right)$$

超几何函数积分表示与部分性质

$$F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tx)^{-a} dt.$$

$$F(\alpha, \beta, \gamma; x) = F(\beta, \alpha, \gamma; x),$$

$$F(\alpha, \beta, \gamma; x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma; x),$$

$$F(\alpha, \beta, \gamma; x) = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma; \frac{x}{x-1}\right),$$

$$F(\alpha, \beta, \gamma; x) = (1-x)^{-\beta} F\left(\beta, \gamma-\alpha, \gamma; \frac{x}{x-1}\right).$$

$(r/a)^n \exp(i*mf)$ 的展开

$$\left(\frac{r}{a}\right)^n \exp(imf) = \sum_{p=-\infty}^{\infty} X_p^{n,m}(e) \exp(ipM)$$

$$X_p^{n,m}(e) = \int_0^{2\pi} \left(\frac{r}{a}\right)^n \cos(mf - pM) dM$$

汉森系数

$$X_p^{n,m}(e) = (1 + \beta)^{-(n+1)} \sum_{q=-\infty}^{+\infty} J_q(pe) X_{p,q}^{n,m}$$

$$\beta = \frac{1}{e} (1 - \sqrt{1 - e^2}) = \frac{e}{1 + \sqrt{1 - e^2}}$$

$$X_{p,q}^{n,m} = \begin{cases} (-\beta)^{(p-m)-q} \binom{n-m+1}{p-m-q} F(p-q-n-1, \\ -m-n-1, p-m-q+1, \beta^2) & (q \leq p-m) \\ (-\beta)^{q-(p-m)} \binom{n+m+1}{q-p+m} F(q-p-n-1, \\ m-n-1, q-p+m+1, \beta^2) & (q \geq p-m) \end{cases}$$

主要内容

- 天体力学概要
- 首次积分
- 椭圆运动展开式
- 无奇点根数、两行根数与正则根数描述
- 轨道空间几何

无奇点根数、正则共轭根数

小偏心率问题、小倾角根数和小偏心率小倾角根数

$$a, e_x = e \cos \omega, e_y = e \sin \omega, i, \Omega, \tilde{\alpha} = \omega + M$$

$$a, e, \tilde{\omega} = \omega + \Omega, h_x = 2 \sin \frac{i}{2} \cos \Omega, h_y = 2 \sin \frac{i}{2} \sin \Omega, M$$

$$a, e \cos \tilde{\omega}, e \sin \tilde{\omega}, 2 \sin \frac{i}{2} \cos \Omega, 2 \sin \frac{i}{2} \sin \Omega, \lambda \quad \tilde{\omega} = \omega + \Omega \quad \tilde{\lambda} = \omega + \Omega + M$$

Delaunay 根数

$$\begin{cases} L = \sqrt{\mu a}, l = M \\ G = L \sqrt{1 - e^2}, g = \omega \\ H = G \cos i, h = \Omega \end{cases}$$

$$F = -K = \frac{\mu^2}{2L^2} + R$$

$$\begin{cases} \dot{p}_i = \frac{\partial F}{\partial q_i} \\ \dot{q}_i = -\frac{\partial F}{\partial p_i} \end{cases}$$

两行根数

AAAAAAAAAAAAAAAAAAAAAAAAA

1 NNNNU NNNNAAA NNNN.NNNNNNNN +.NNNNNNNN +NNNN-N +NNNN-N N NNNNN

2 NNNN NNN.NNNN NNN.NNNN NNNNNNN NNN.NNNN NNN.NNNN NN.NNNNNNNNNNNNN

Field	Column	Description
1.1	01	Line Number of Element Data
1.2	03-07	Satellite Number
1.3	08	Classification
1.4	10-11	International Designator (Last two digits of launch year)
1.5	12-14	International Designator (Launch number of the year)
1.6	15-17	International Designator (Piece of the launch)
1.7	19-20	Epoch Year (Last two digits of year)
1.8	21-32	Epoch (Day of the year and fractional portion of the day)
1.9	34-43	First Time Derivative of the Mean Motion
1.10	45-52	Second Time Derivative of Mean Motion (decimal point assumed)
1.11	54-61	BSTAR drag term (decimal point assumed)
1.12	63	Ephemeris type
1.13	65-68	Element number
1.14	69	Checksum (Modulo 10) (Letters, blanks, periods, plus signs = 0; minus signs = 1)

Field	Column	Description
2.1	01	Line Number of Element Data
2.2	03-07	Satellite Number
2.3	09-16	Inclination [Degrees]
2.4	18-25	Right Ascension of the Ascending Node [Degrees]
2.5	27-33	Eccentricity (decimal point assumed)
2.6	35-42	Argument of Perigee [Degrees]
2.7	44-51	Mean Anomaly [Degrees]
2.8	53-63	Mean Motion [Revs per day]
2.9	64-68	Revolution number at epoch [Revs]
2.10	69	Checksum (Modulo 10)

两行根数算例

16 Sep 2019 01:06:42

Satellite-JASON-2_33105

Two Line Element Set

```
-----  
1 33105U 08032A 17088.90414795 -.00000066 00000-0 -20983-4 0 9998  
2 33105 66.0401 286.3042 0007614 274.4658 183.4887 12.80932272410381  
-----
```

16 Sep 2019 01:01:39

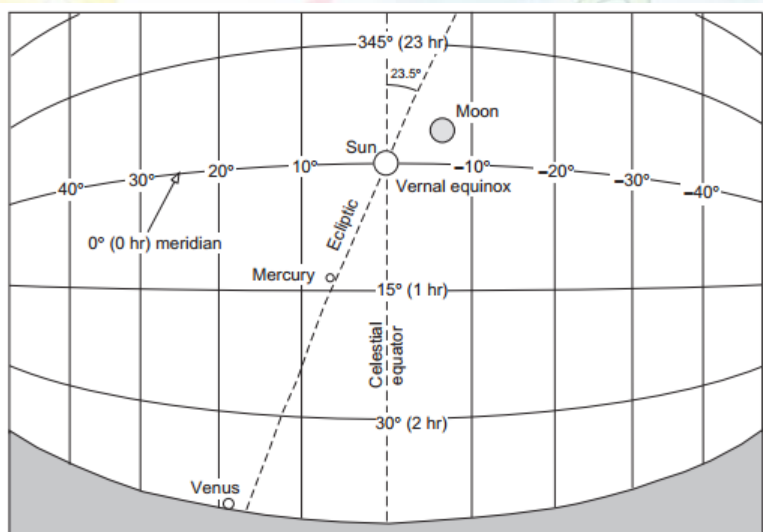
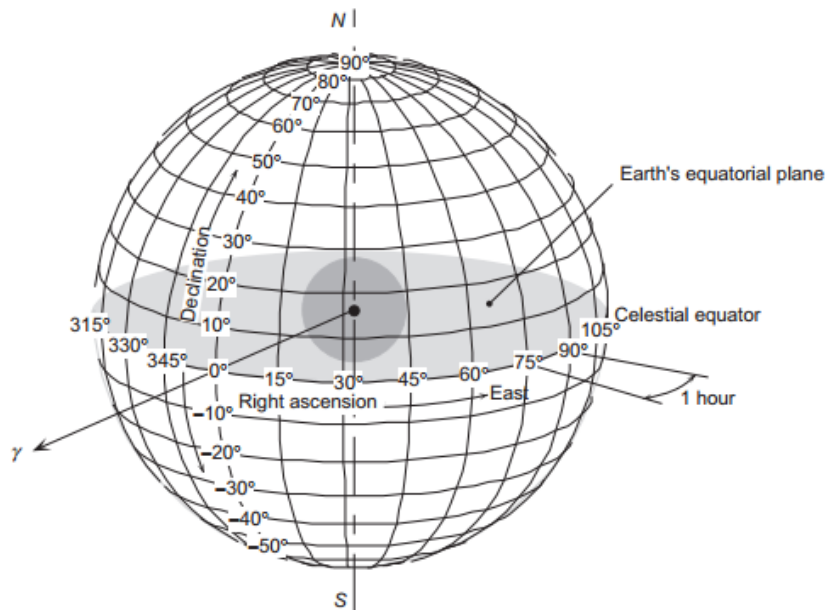
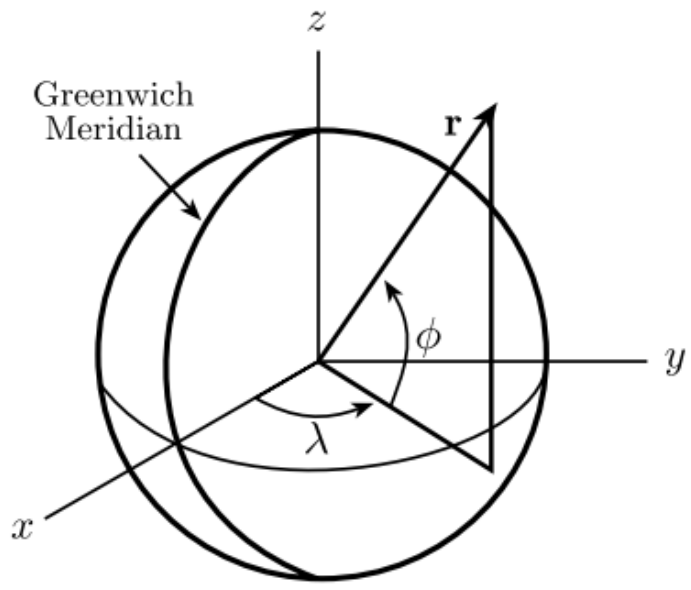
Satellite-JASON-2_33105: J2000 Position & Velocity

Time (UTCG)	x (km)	y (km)	z (km)	vx (km/sec)	vy (km/sec)	vz (km/sec)
16 Sep 2019 04:00:00.000	-5291.777394	-845.038485	-5558.116835	-3.472599	-4.820868	4.034093
16 Sep 2019 04:01:00.000	-5491.791680	-1132.825631	-5307.530043	-3.192813	-4.769552	4.316635
16 Sep 2019 04:02:00.000	-5674.714794	-1417.087258	-5040.388879	-2.903040	-4.703381	4.585762
16 Sep 2019 04:03:00.000	-5839.974295	-1696.937917	-4757.523749	-2.604178	-4.622557	4.840631
16 Sep 2019 04:04:00.000	-5987.052414	-1971.505544	-4459.814330	-2.297151	-4.527326	5.080443
16 Sep 2019 04:05:00.000	-6115.487742	-2239.934180	-4148.186898	-1.982912	-4.417977	5.304444
16 Sep 2019 04:06:00.000	-6224.876737	-2501.386655	-3823.611511	-1.662437	-4.294847	5.511930
16 Sep 2019 04:07:00.000	-6314.875062	-2755.047226	-3487.099033	-1.336723	-4.158314	5.702244

主要内容

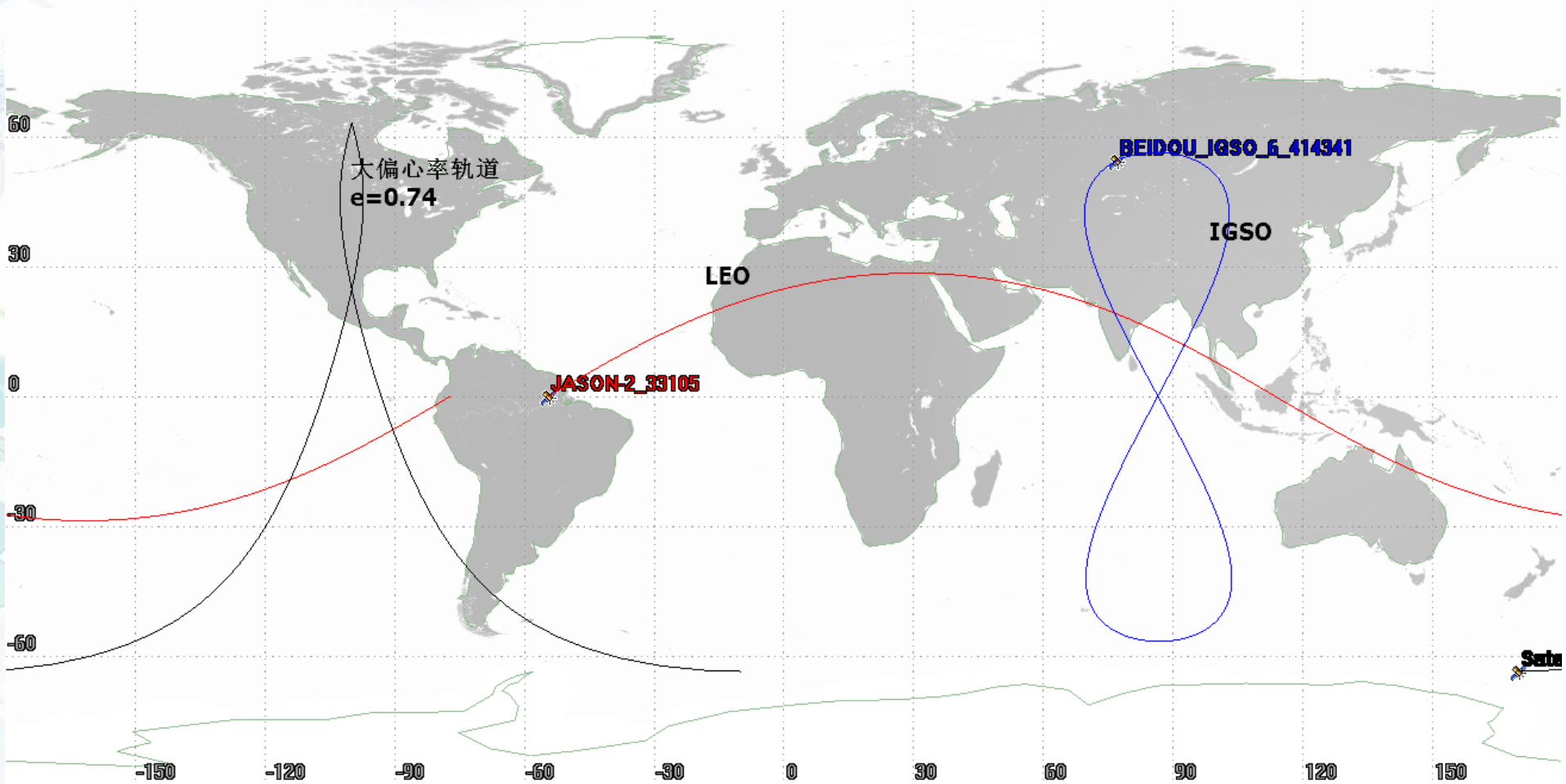
- 天体力学概要
- 首次积分
- 椭圆运动展开式
- 无奇点根数、两行根数与正则根数描述
- 轨道空间几何

星下点轨迹



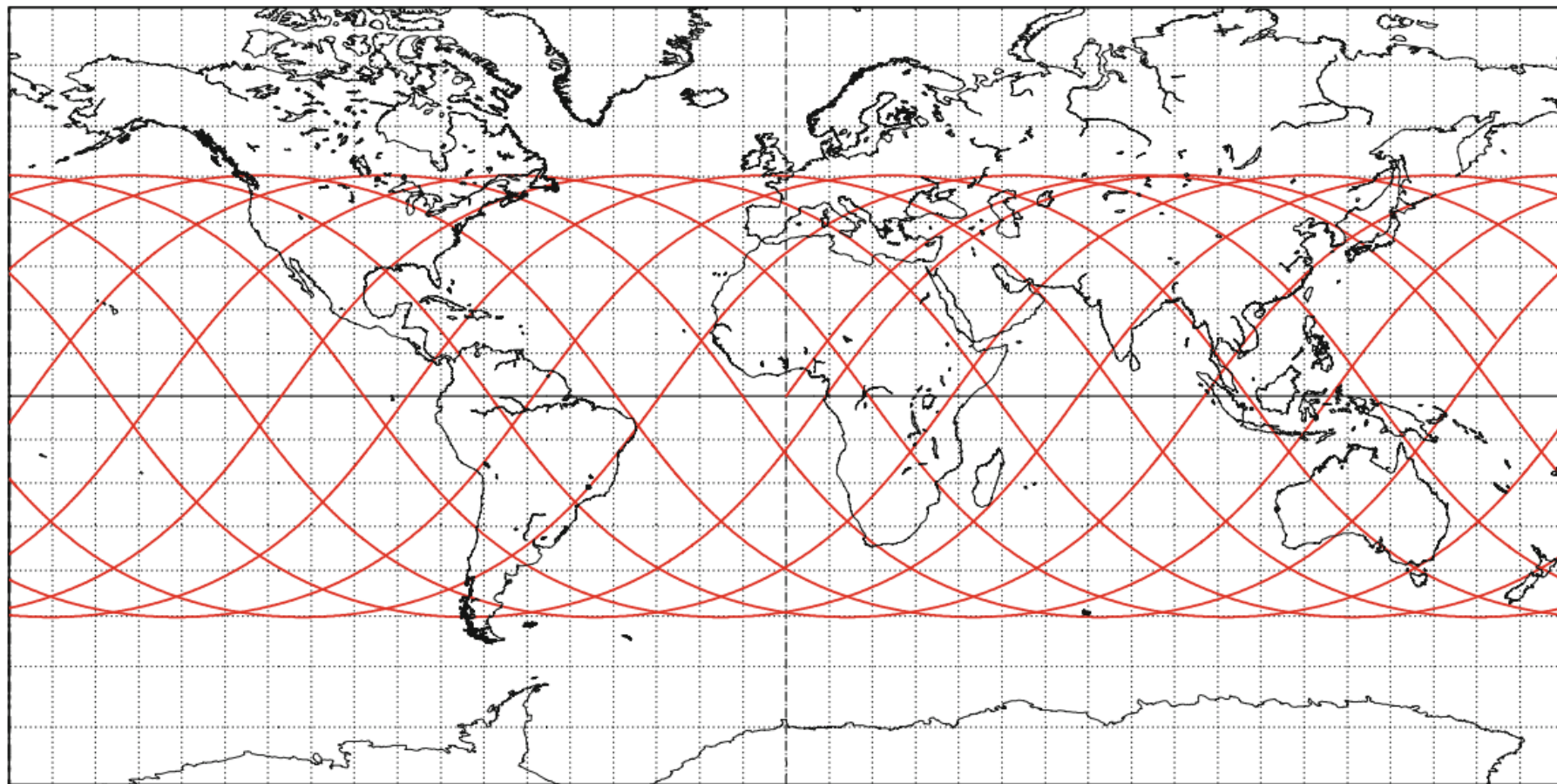
卫星星下点轨迹是星下点在地球表面通过的路径。星下点地理纬度为卫星赤纬，地理经度为卫星赤经与t时刻格林尼治恒星时之差。(satellite orbit)

IGSO、LEO与大偏心率轨道星下点

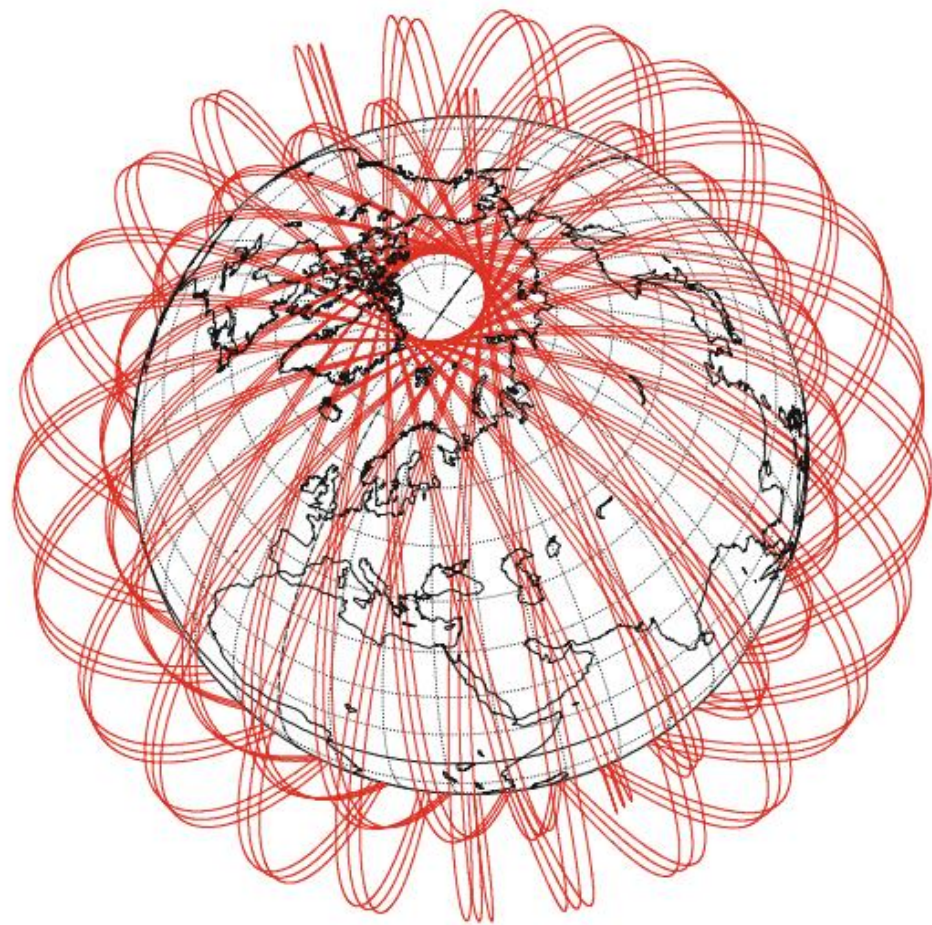
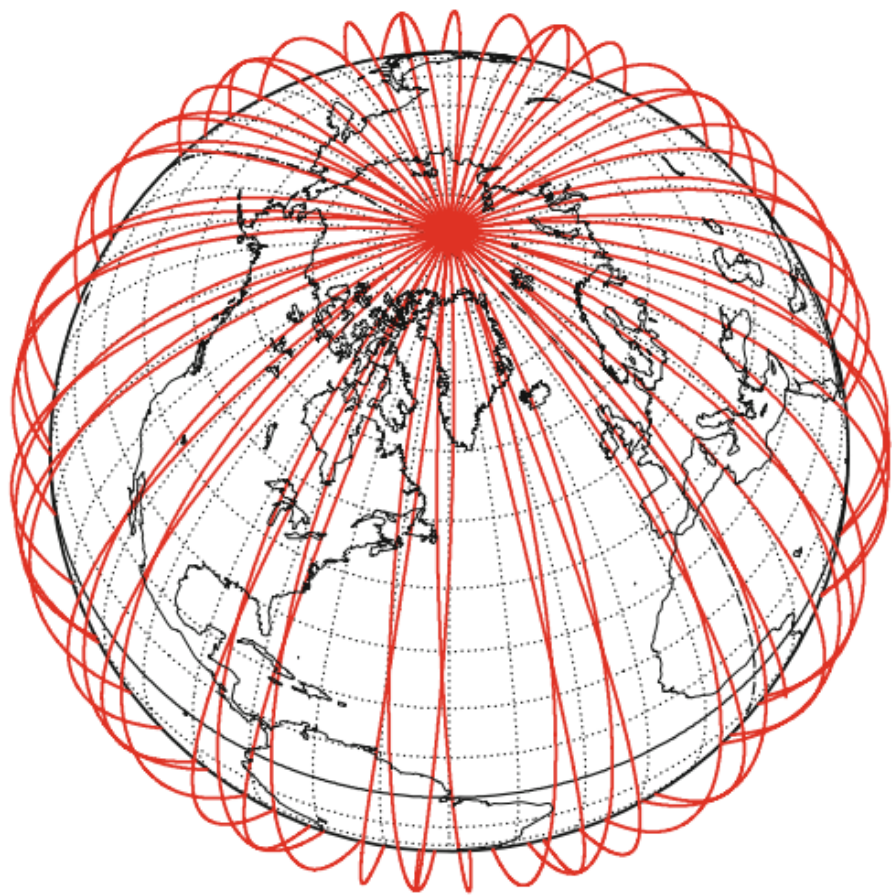


多圈星下点

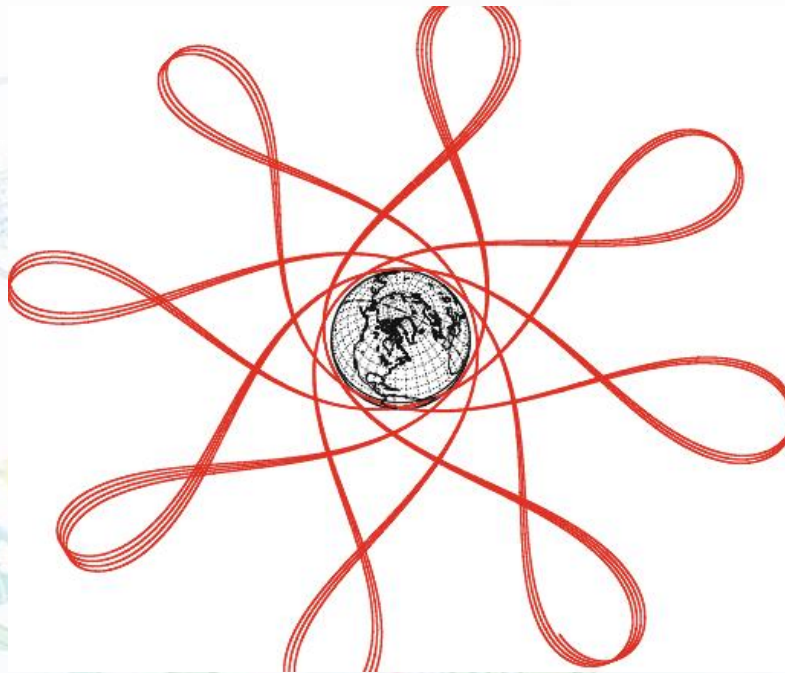
Altitude=1488.5 km,
Period = 115.65 min
Inclination=50.01 deg



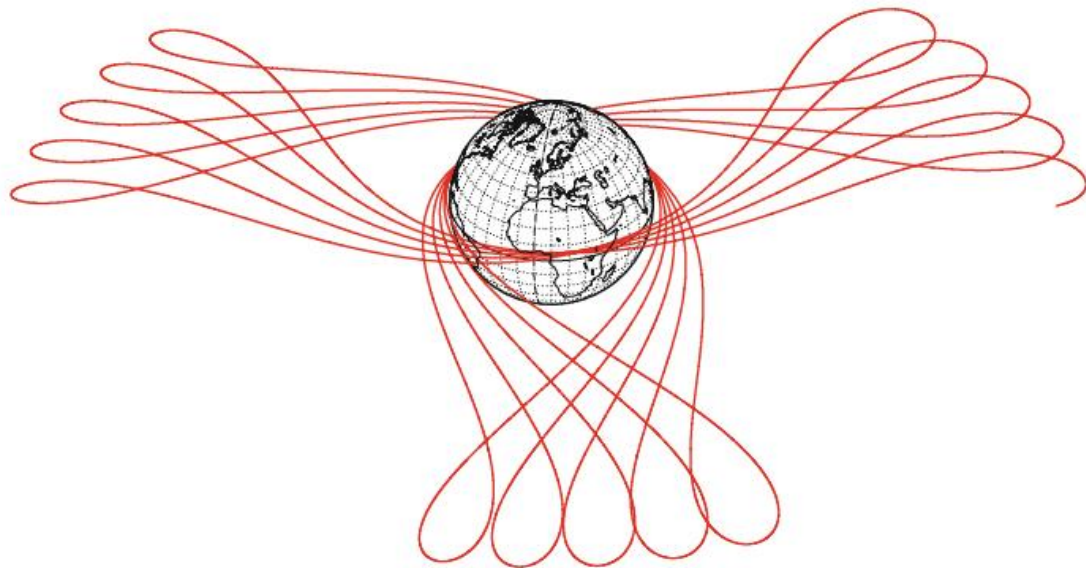
空间视角



大偏心率轨道



Equiv. altit. = 15559.4 km $a = 21937.541$ km
Inclination = 9.95° $e = 0.682033$
Period = 538.26 min * rev./day = 2.68
 $h_a = 30522$ km; $h_p = 597$ km; arg. perigee: $+24.96^\circ$



Equiv. altit. = 14233.5 km $a = 20611.604$ km
Inclination = 7.15° $e = 0.679397$
Period = 490.12 min * Revol./d. = 2.94
 $h_a = 28237$ km ; $h_p = 230$ km ; arg. perigee: $+55.73^\circ$



Q&A!