中国科学院上海天文台

中国科学院大学

University of Chinese Academy of Sciences

空间飞行器精密定轨

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- 宇宙学中的动力学问题
- 引力论与后牛顿天体力学
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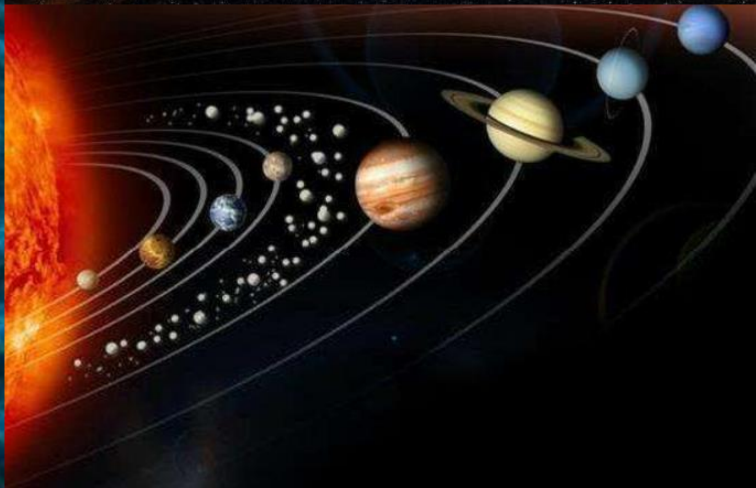
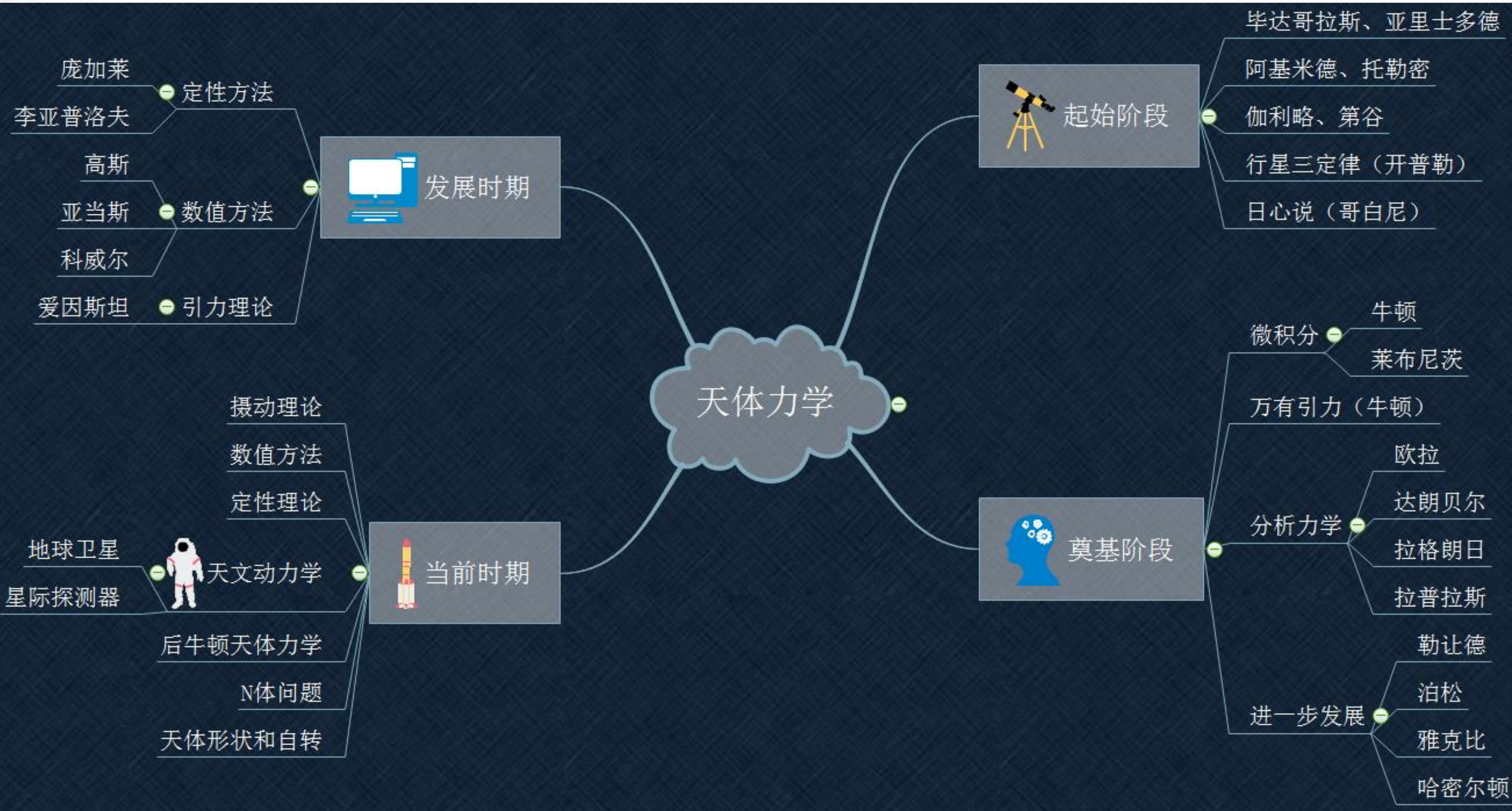


天体力学的发展


天体力学
起始阶段
毕达哥拉斯、亚里士多德
阿基米德、托勒密
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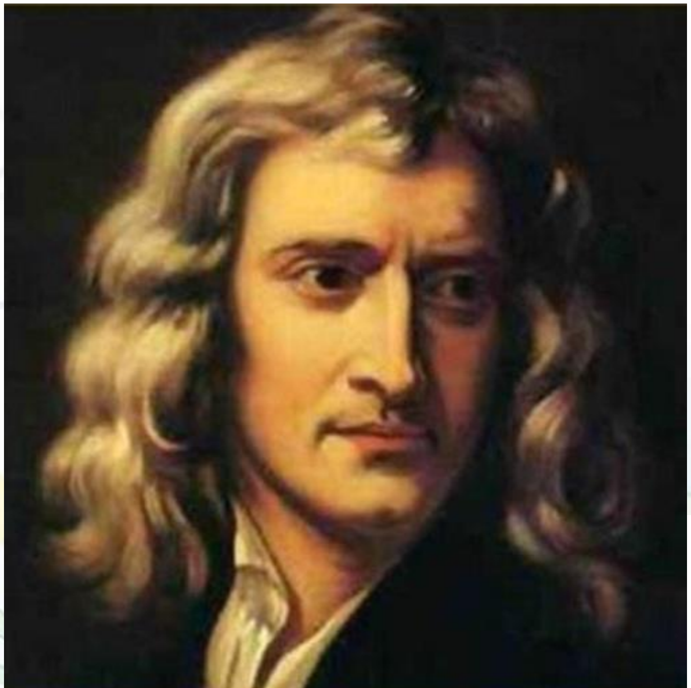
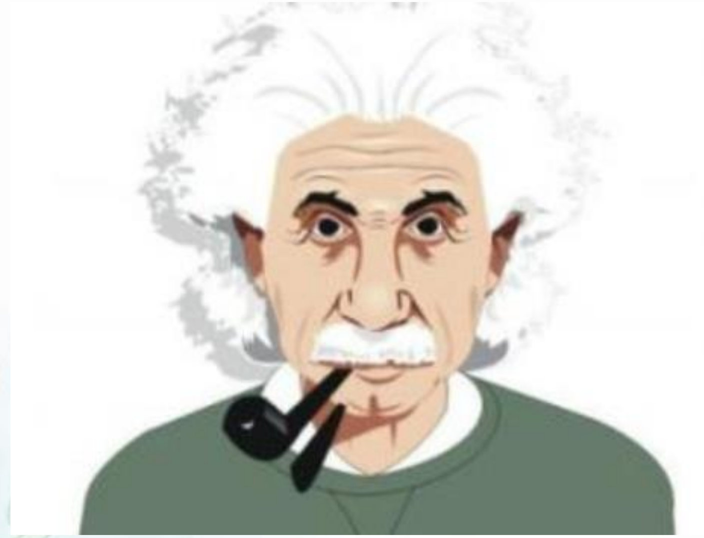
天体形状和自转

万有引力与参数化后牛顿引力方程





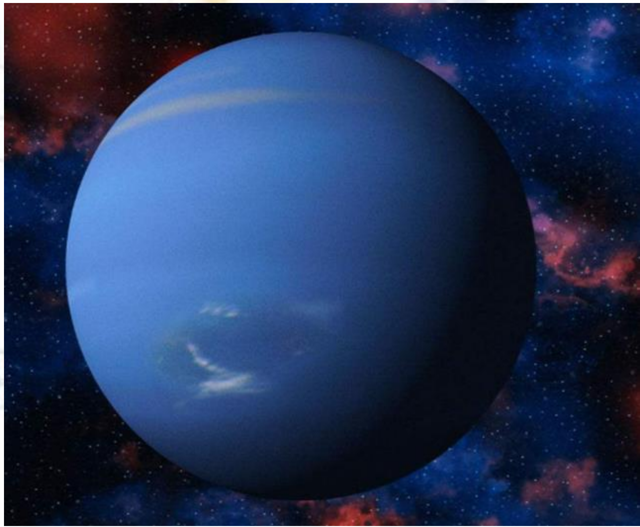
$$
\begin{aligned}
\ddot{\boldsymbol{x}}_i = & -k^2 \sum_{j=0,j\neq i}^{n} m_j \frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|^3}\Bigg\{1 - \frac{2\,k^2\,(\beta+\gamma)}{c^2}\sum_{k=0,k\neq i}^{n} \frac{m_k}{|\boldsymbol{x}_i-\boldsymbol{x}_k|} \\
& - \frac{k^2\,(2\beta-1)}{c^2}\sum_{k=0,k\neq j}^{n} \frac{m_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|} + \gamma\,\frac{\dot{\boldsymbol{x}}_i^2}{c^2} + (1+\gamma)\,\frac{\dot{\boldsymbol{x}}_j^2}{c^2} \\
& - \frac{2\,(1+\gamma)}{c^2}\,\dot{\boldsymbol{x}}_i\cdot\dot{\boldsymbol{x}}_j - \frac{3}{2\,c^2}\left[\frac{(\boldsymbol{x}_i-\boldsymbol{x}_j)\,\dot{\boldsymbol{x}}_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|}\right]^2 - \frac{1}{2\,c^2}\,(\boldsymbol{x}_i-\boldsymbol{x}_j)\cdot\ddot{\boldsymbol{x}}_j\Bigg\} \\
& + \frac{k^2}{c^2}\sum_{j=0,j\neq i}^{n} \frac{m_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|^3}\Big\{(\boldsymbol{x}_i-\boldsymbol{x}_j)\big[\,(2+2\,\gamma)\,\dot{\boldsymbol{x}}_i \\
& - (1+2\,\gamma)\,\dot{\boldsymbol{x}}_j\,\big]\Big\}\cdot(\dot{\boldsymbol{x}}_i-\dot{\boldsymbol{x}}_j) + \frac{k^2\,(3+4\,\gamma)}{2\,c^2}\sum_{j=0,j\neq i}^{n} m_j\,\frac{\ddot{\boldsymbol{x}}_j}{|\boldsymbol{x}_i-\boldsymbol{x}_j|}\,,
\end{aligned}
$$

天体力学的重大胜利：哈雷彗星的回归与海王星的发现

哈雷在1705年发表了《彗星天文学论说》，宣布1682年曾引起世人极大恐慌的大彗星将于1758年再次出现于天空（后来他估计到木星可能影响到它的运动时，把回归的日期推迟到1759年）。当时哈雷已年过五十，知道在有生之年无缘再见到这颗大彗星了。哈雷去世10多年后，1758年底，这颗第一个被预报回归的彗星被一位业余天文学家观测到了。公元前240年起的每次回归我国都有所记载，最早的一次可能是周武王伐纣之年，即公元前1057年。哈雷彗星每隔大约76年都会按时回归。最近一次是1986年。

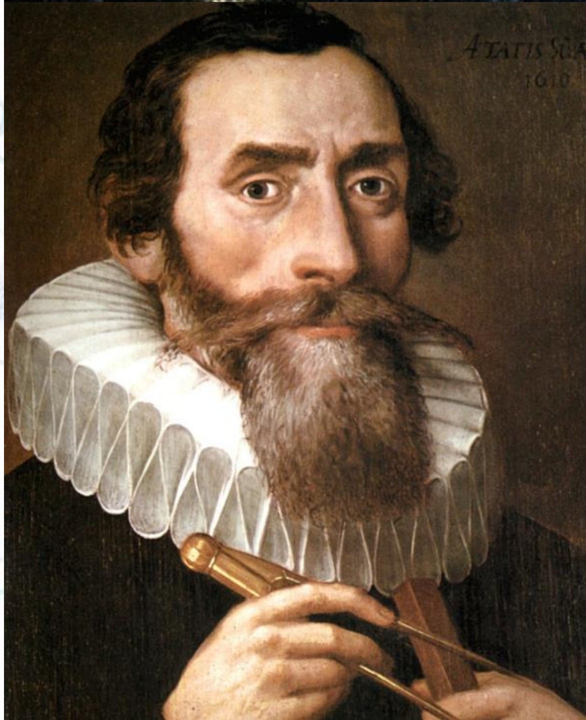
在1821年，Alexis Bouvard出版了天王星的轨道表，随后的观测显示出与表中的位置有越来越大的偏差，使得布瓦尔假设有一个摄动体存在。勒维耶完成了海王星位置的推算。

在1846年9月23日晚间，海王星被发现了，与勒维耶预测的位置相距不到1°。



开普勒行星三定律

- 第一定律：各行星的轨道均为椭圆，太阳位于该椭圆的一个焦点上。
- 第二定律：行星与太阳的连线在相等时间内扫过的面积相等。
- 第三定律：行星轨道周期的平方与行星至太阳平均距离三次方成正比。



二体方程

定义中心天体势

$$\phi = -G\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

$$F(\mathbf{r}) = G\int \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}')d^3\mathbf{r}' = -\nabla\phi_{\mathbf{r}}$$

$$\nabla^2\phi = 4\pi G\rho$$

$$\nabla \bullet F(\mathbf{r}) = -G\rho(\mathbf{r})\int_{|\mathbf{r}-\mathbf{r}'|=h} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^2S'$$

$$\nabla^2\phi = 4\pi G\rho$$ 泊松方程

牛顿第一定理：球壳对位于其内部任意一点上的物体的引力之和为零。

牛顿第二定理：闭合球壳对位于球壳外任一物体的引力，等于把球壳所有质量集中于球壳中心上的点质量对该物体的引力。

因此把近球形（密度较为规则）大天体当作质点是一个很好的近似。

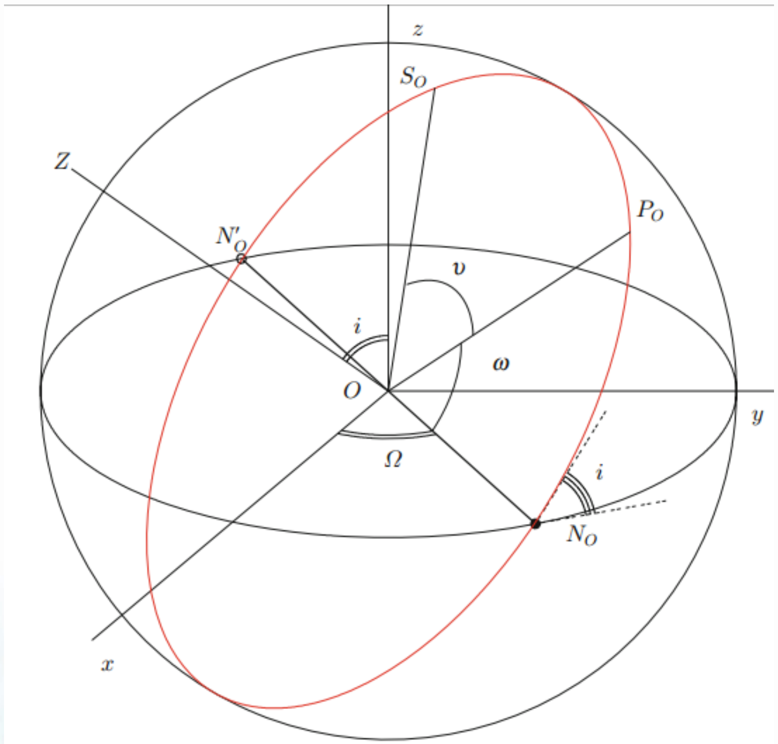
$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}$$

动量矩积分

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} \qquad \boldsymbol{r}\times\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r}\times\boldsymbol{r} = 0$$

$$\frac{d(\boldsymbol{r}\times\dot{\boldsymbol{r}})}{dt} = 0 \qquad \boldsymbol{r}\times\dot{\boldsymbol{r}} = \text{constant}$$

$$\mathbf{R} = \frac{\mathbf{r}\times\mathbf{v}}{h} = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = \begin{pmatrix} \sin i \sin\Omega \\ -\sin i \cos\Omega \\ \cos i \end{pmatrix}$$



令该常数向量为 $\boldsymbol{h}$，即为卫星的动量矩。由于 $\boldsymbol{r}$ 和 $\dot{\boldsymbol{r}}$ 始终垂直于 $\boldsymbol{h}$。因此其运动始终位于一个平面内，即轨道平面。动量矩 $\boldsymbol{h}$ 的方向与卫星轨道面法线是平行的，称 $\boldsymbol{h}$ 和坐标轴 Z 轴的夹角为轨道倾角 i。轨道平面和赤道面的交现为节线 ON，节点为轨道升交点，节线 ON 与 X 轴的的夹角称为升交点赤经 Ω。轨道面倾角和升交点赤经决定了轨道运动平面的空间定向。

正交曲线坐标表示的运动

笛卡尔坐标和正交曲线坐标之间的关系（其中,x,y,z分别是q1,q2,q3二阶连续可微函数）

$$\boldsymbol{r}=\boldsymbol{r}\left(q_1,q_2,q_3\right)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$$

$$\boldsymbol{e}_i=\frac{1}{H_i}\frac{\partial\boldsymbol{r}}{\partial q_i},\quad \frac{\partial\boldsymbol{r}}{\partial q_i}=\frac{\partial x}{\partial q_i}\boldsymbol{i}+\frac{\partial y}{\partial q_i}\boldsymbol{j}+\frac{\partial z}{\partial q_i}\boldsymbol{k}$$

$$H_i=\left|\frac{\partial\boldsymbol{r}}{\partial q_i}\right|=\sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2}$$

拉梅系数

$$\boldsymbol{v}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\frac{\partial\boldsymbol{r}}{\partial q_1}\dot{q}_1+\frac{\partial\boldsymbol{r}}{\partial q_2}\dot{q}_2+\frac{\partial\boldsymbol{r}}{\partial q_3}\dot{q}_3=v_{q_1}\boldsymbol{e}_1+v_{q_2}\boldsymbol{e}_2+v_{q_3}\boldsymbol{e}_3$$

$$v_{q_i}=H_i\dot{q}_i$$

正交曲线坐标系加速度表达式

$$w_{q_i} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{e}_i = \frac{1}{H_i}\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i}\right) = \frac{1}{H_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v} \cdot \frac{\partial \boldsymbol{r}}{\partial q_i}\right) - \boldsymbol{v} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}}{\partial q_i}\right)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}}{\partial q_i}\right) = \frac{\partial^2 \boldsymbol{r}}{\partial q_i \partial q_1}\dot{q}_1 + \frac{\partial^2 \boldsymbol{r}}{\partial q_i \partial q_2}\dot{q}_2 + \frac{\partial^2 \boldsymbol{r}}{\partial q_i \partial q_3}\dot{q}_3$$

$$\frac{\partial \boldsymbol{v}}{\partial q_i} = \frac{\partial^2 \boldsymbol{r}}{\partial q_1 \partial q_i}\dot{q}_1 + \frac{\partial^2 \boldsymbol{r}}{\partial q_2 \partial q_i}\dot{q}_2 + \frac{\partial^2 \boldsymbol{r}}{\partial q_3 \partial q_i}\dot{q}_3$$

$$w_{q_i} = \frac{1}{H_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}_i}\right) - \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial q_i}\right]$$

代入 $T = v^2/2$

$$w_{q_i} = \frac{1}{H_i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right)$$

柱坐标与球坐标速度与加速度表达式

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z; \qquad H_r = 1, \quad H_\varphi = r, \quad H_z = 1$$

$$v_r = \dot{r}, \quad v_\varphi = r\dot{\varphi}, \quad v_z = \dot{z};$$

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2);$$

$$w_r = \ddot{r} - r\dot{\varphi}^2, \quad w_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi}, \quad w_z = \ddot{z}.$$

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta; \quad H_r = 1, \quad H_\varphi = r\sin\theta, \quad H_\theta = r;$$

$$v_r = \dot{r}, \quad v_\varphi = r\sin\theta\dot{\varphi}, \quad v_\theta = r\dot{\theta};$$

$$T = \frac{1}{2}(\dot{r}^2 + r^2\sin^2\theta\dot{\varphi}^2 + r^2\dot{\theta}^2);$$

$$w_r = \ddot{r} - r\sin^2\theta\dot{\varphi}^2 - r\dot{\theta}^2, \quad w_\varphi = r\sin\theta\ddot{\varphi} + 2\sin\theta\dot{r}\dot{\varphi} + 2r\cos\theta\dot{\varphi}\dot{\theta},$$

$$w_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2.$$

平面内运动（极坐标）

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\dfrac{\mu}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

$$r^2\dot{\theta} = \text{constant}$$

面积积分的标量形式

用u=1/r代替r,可以求出r关于theta的导函数

$$\begin{cases} \dot{r} = \dfrac{dr}{dt} = \dfrac{dr}{d\theta}\dfrac{d\theta}{dt} = \dfrac{d}{d\theta}\left(\dfrac{1}{u}\right)\dfrac{d\theta}{dt} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}\dot{\theta} = -h\dfrac{du}{d\theta} \\ \ddot{r} = \dfrac{d\dot{r}}{dt} = \dfrac{d}{dt}\left(-h\dfrac{d\mu}{d\theta}\right) = \dfrac{d}{d\theta}\left(-h\dfrac{du}{d\theta}\right)\dot{\theta} = -h^2u^2\dfrac{d^2u}{d\theta^2} \end{cases}$$

开普勒第一定律：椭圆方程

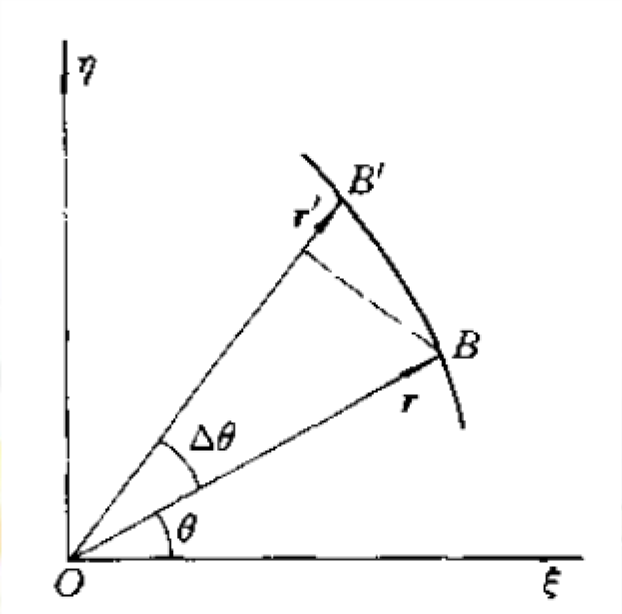
$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

$$\frac{1}{r} = \frac{\mu}{h^2}\left[1 + e\cos(\theta - \omega)\right]$$

$$r = \frac{p}{1 + e\cos(\theta - \omega)}$$

e 和 ω 为新的积分常数

平面内运动



$$\Delta A = \frac{1}{2} r r' \sin \Delta\theta$$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r r' \frac{\Delta\theta}{\Delta t} \frac{\sin \Delta\theta}{\Delta\theta}$$

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h$$

开普勒第二定律

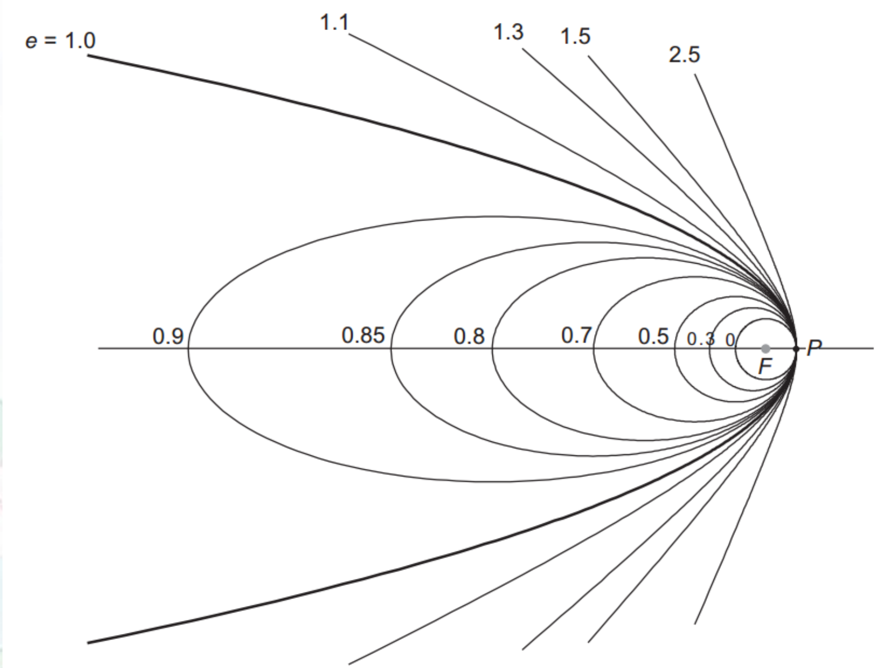
卫星一个周期T内，扫过的面积为pi *a*b，有 $\pi ab = \frac{1}{2} hT$

$$a = \frac{h^2}{\mu\left(1-e^2\right)}, a\left(1-e^2\right) = b\sqrt{1-e^2}, e = \sqrt{1-\frac{b^2}{a^2}}$$

开普勒第三定律 $T = 2\pi\sqrt{\frac{a^3}{\mu}}$

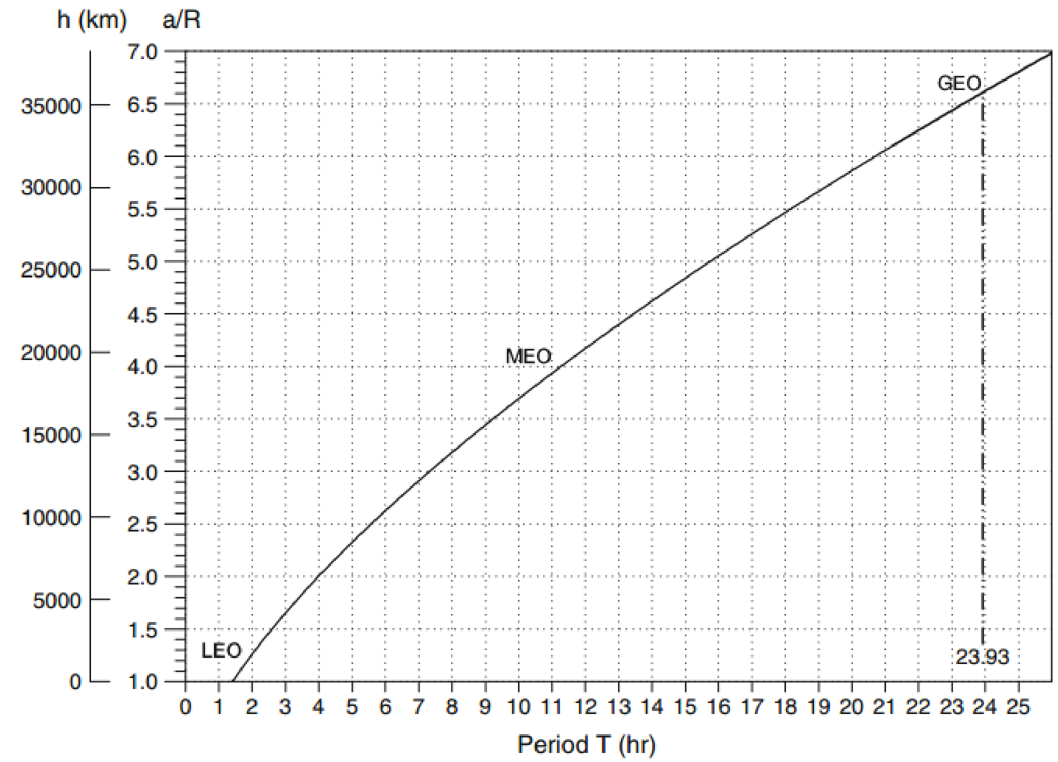
$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

抛物与双曲轨道



Type	Eccentricity	Semi-latus rectum	Perihelion	Energy
Circle	$e = 0$	$p = a$	a	$-\frac{\mu}{2a} < 0$
Ellipse	$e < 1$	$p = a(1 - e^2)$	$a(1 - e)$	$-\frac{\mu}{2a} < 0$
Parabola	$e = 1$	p	$q = \frac{p}{2}$	$= 0$
Hyperbola	$e > 1$	$p = a(e^2 - 1)$	$a(e - 1)$	$+\frac{\mu}{2a} > 0$

地球卫星近圆轨道高度与周期的关系




```
song@SONG-PC F:\test
$ satperiod  400    20
     a(km)  Height(km)    T(minute)     T(hour)
   6378.14        0.00        84.49        1.41
   6778.14      400.00        92.56        1.54
   7178.14      800.00       100.87        1.68
   7578.14     1200.00       109.42        1.82
   7978.14     1600.00       118.20        1.97
   8378.14     2000.00       127.20        2.12
   8778.14     2400.00       136.42        2.27
   9178.14     2800.00       145.85        2.43
   9578.14     3200.00       155.48        2.59
   9978.14     3600.00       165.32        2.76
  10378.14     4000.00       175.36        2.92
  10778.14     4400.00       185.60        3.09
  11178.14     4800.00       196.03        3.27
  11578.14     5200.00       206.64        3.44
  11978.14     5600.00       217.44        3.62
  12378.14     6000.00       228.42        3.81
  12778.14     6400.00       239.59        3.99
  13178.14     6800.00       250.92        4.18
  13578.14     7200.00       262.43        4.37
  13978.14     7600.00       274.12        4.57
  14378.14     8000.00       285.97        4.77
```

开普勒方程

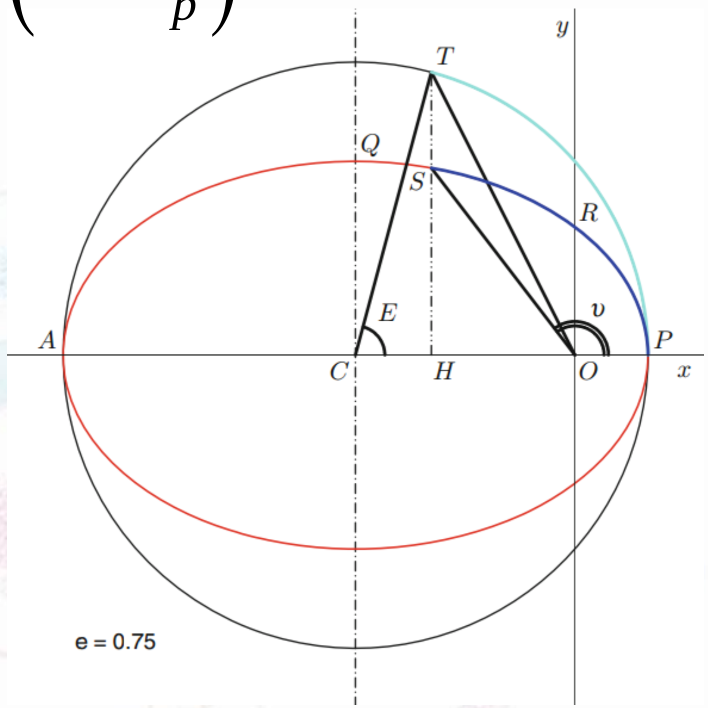
$$r^2\dot{\theta} = h \qquad \int_0^f r^2 df = h\left(t - t_p\right)$$

$$\dot{\theta} = \dot{f}$$

$$t - t_p = \frac{h^3}{\mu^2}\int_0^f \frac{df}{\left(1 + e\cos f\right)^2}$$

$$= \frac{h^2}{\mu^2}\int_0^E \frac{1 - e\cos E}{\left(1 - e^2\right)^{\frac{3}{2}}} dE$$

$$= \sqrt{\frac{a^3}{\mu}}\left(E - e\sin E\right)$$



$$n\left(t - t_p\right) = M = E - e\sin E$$

t_p 是第六个积分常数

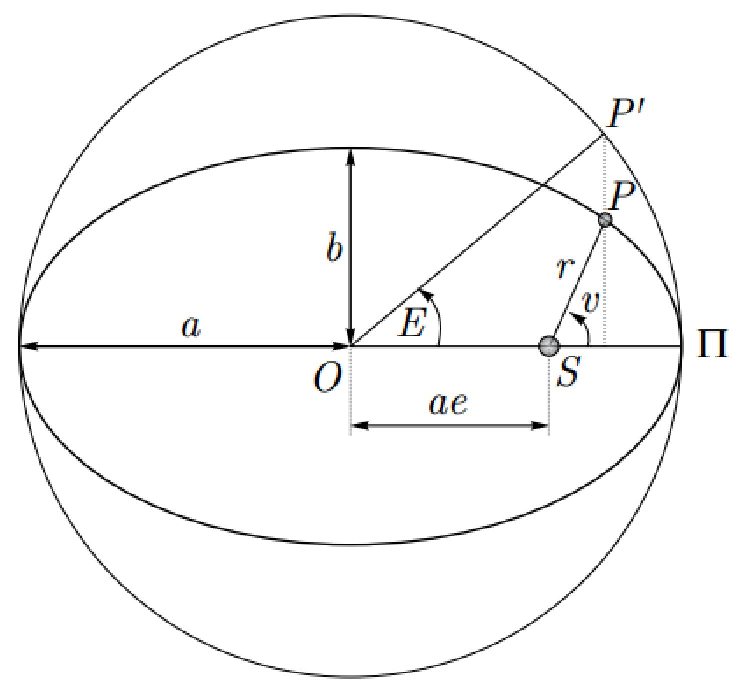
真近点角、偏近点角与平近点角

$$\sin f = \frac{(1-e^2)^{1/2}\sin E}{1-e\cos E} \qquad \sin E = \frac{(1-e^2)^{1/2}\sin f}{1+e\cos f}$$

$$\cos f = \frac{\cos E - e}{1-e\cos E} \qquad \cos E = \frac{e+\cos f}{1+e\cos f}$$

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{E}{2} \qquad \tan\frac{E}{2} = \left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{f}{2}$$

$$\begin{aligned}
M &= E - e\sin E \quad \text{(Kepler's Equation)} \\
\frac{df}{dE} &= \frac{(1-e^2)^{1/2}}{1-e\cos E} = \frac{1+e\cos f}{(1-e^2)^{1/2}} \\
\frac{dM}{dE} &= 1-e\cos E = \frac{1-e^2}{1+e\cos f} \\
\frac{dM}{df} &= \frac{(1-e\cos E)^2}{(1-e^2)^{1/2}} = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2}\,.
\end{aligned}$$



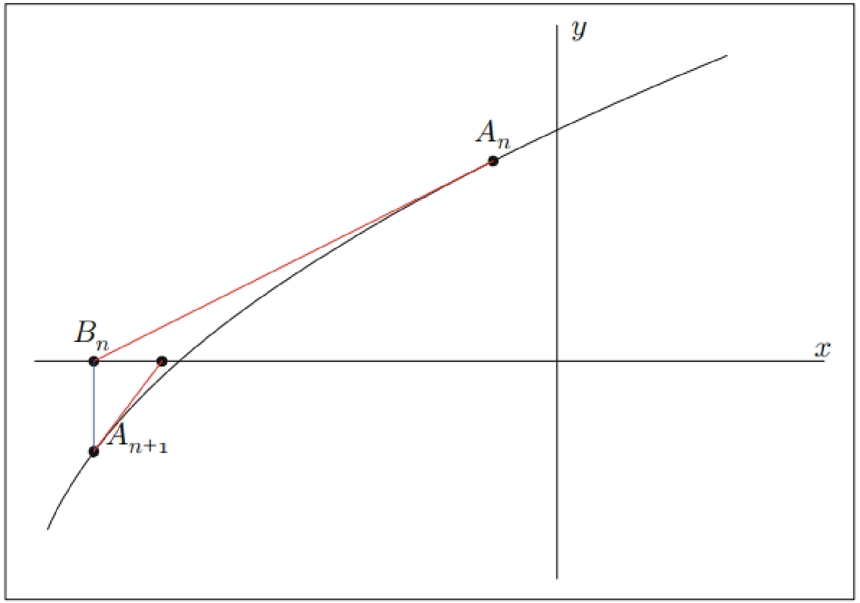
开普勒方程数值解

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(E) = E - e\sin E - M$$

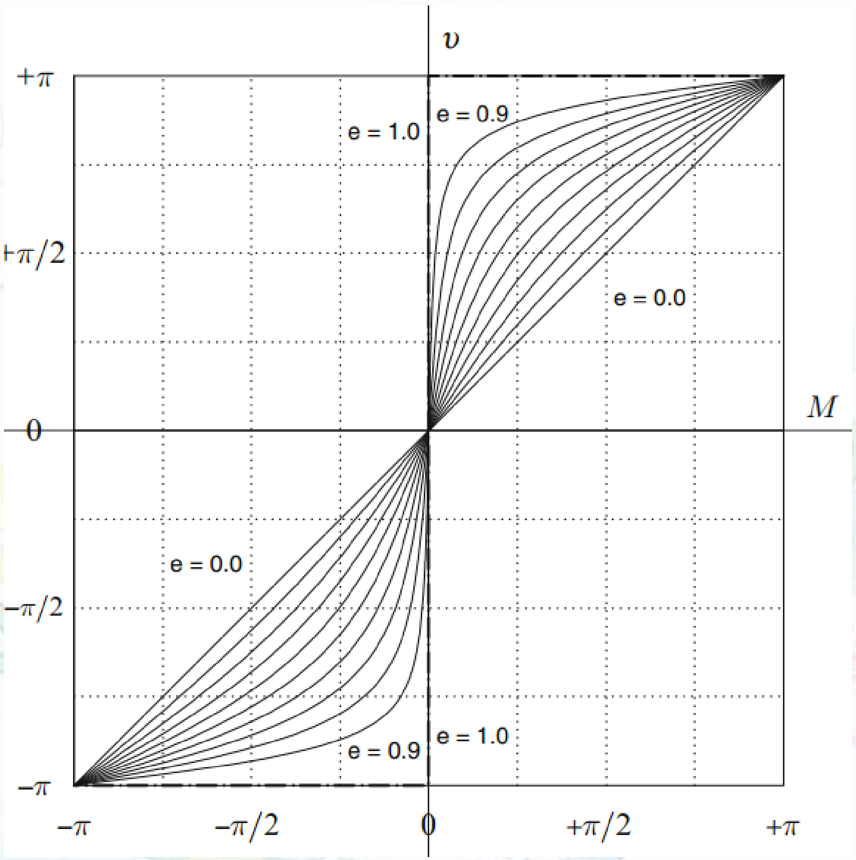
$$f'(E) = 1 - e\cos E$$

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E_n}$$

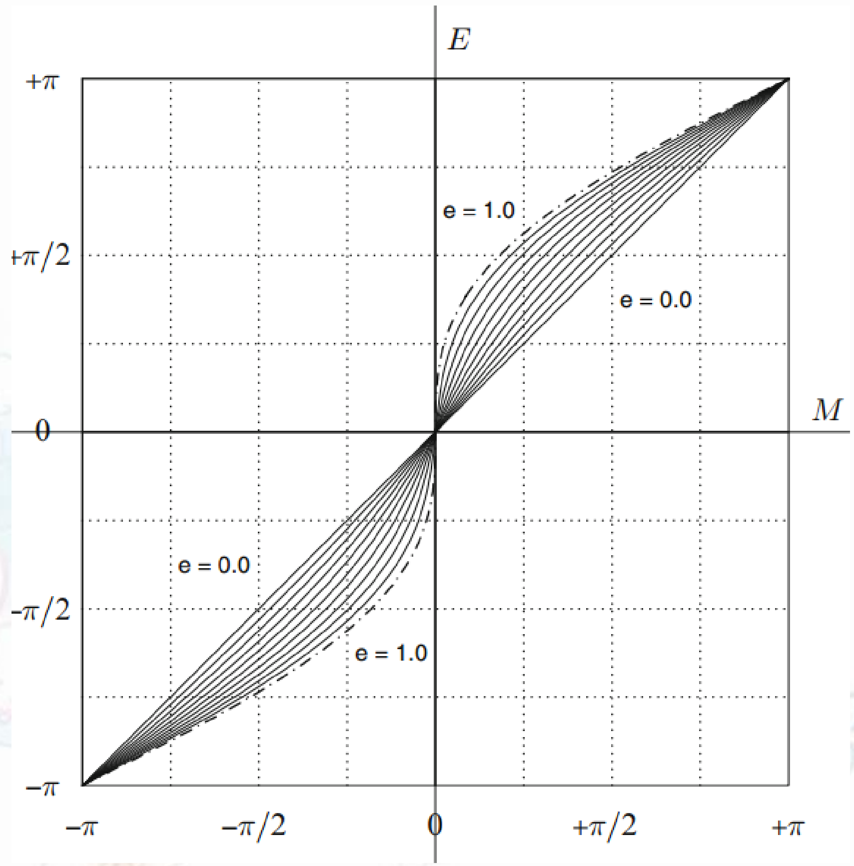


$$v = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right)$$

近点角关系图



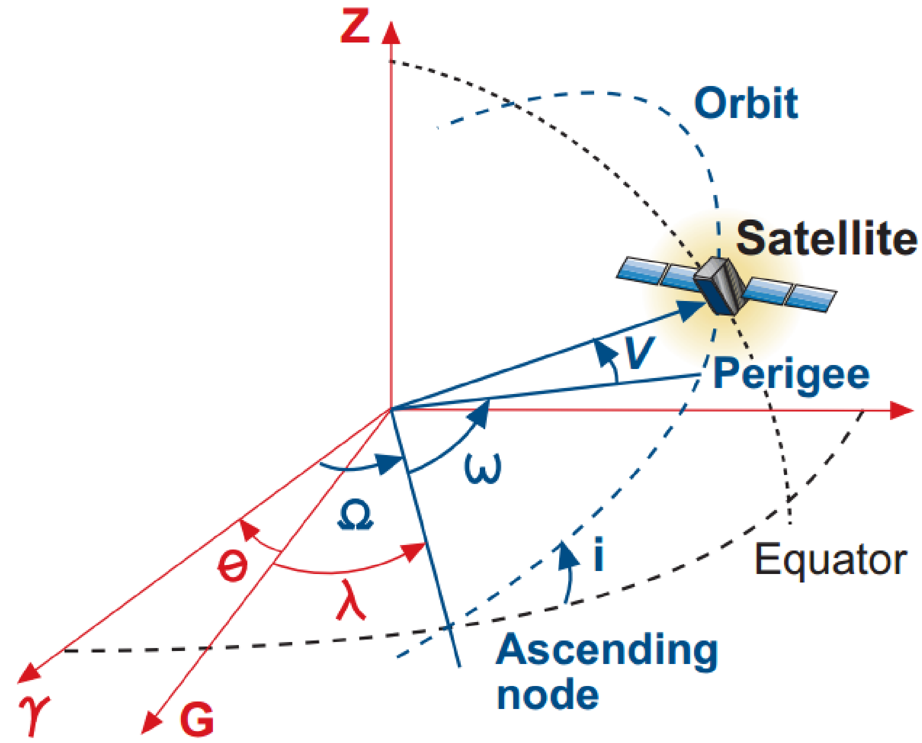
υ
+π
+π/2
0
−π/2
−π
M
e = 1.0
e = 0.9
e = 0.0
e = 0.0
e = 0.9
e = 1.0
−π
−π/2
0
+π/2
+π

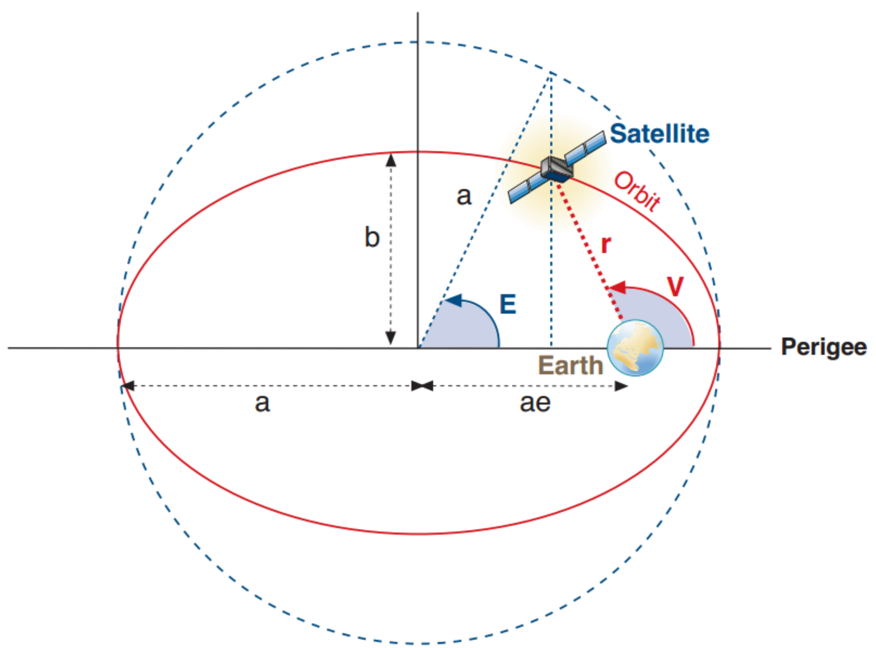


E
+π
+π/2
0
-π/2
−π
M
e = 1.0
e = 0.0
e = 0.0
e = 1.0
−π
−π/2
0
+π/2
+π

开普勒轨道根数

- 形状:
  - a ＝ 半长径
  - e ＝ 偏心率
- 空间定向:
  - i ＝ 轨道倾角
  - Ω ＝ 升交点赤经
  - ω ＝ 近星点幅角
- 位置:
  - ν ＝ 真近点角





根数与位置速度

真近点角、平近点角、偏近点角度及过近星点时刻关系

$$M(t)=n(t-T_0)$$

$$E(t)=M(t)+e\sin E(t)$$

$$\tan\frac{v}{2}=\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{E}{2}$$

$$n=\frac{2\pi}{P}=\sqrt{\frac{\mu}{a^3}}$$

$$\begin{aligned}\vec{r}&=r\hat{r}=r\cos f\hat{P}+r\sin f\hat{Q}\\&=a(\cos E-e)\hat{P}+a\sqrt{1-e^2}\sin E\hat{Q}\end{aligned}$$

$$\begin{aligned}\dot{\vec{r}}&=-\sqrt{\frac{\mu}{p}}\left[\sin f\hat{P}-(\cos f+e)\hat{Q}\right]\\&=-\frac{\sqrt{\mu a}}{r}\left[\sin E\hat{P}-\sqrt{1-e^2}\cos E\hat{Q}\right]\end{aligned}$$

$$\hat{P}=\begin{pmatrix}\cos\Omega\cos\omega-\sin\Omega\sin\omega\cos i\\\sin\Omega\cos\omega+\cos\Omega\sin\omega\cos i\\\sin\omega\sin i\end{pmatrix}$$

$$\hat{Q}=\begin{pmatrix}-\cos\Omega\sin\omega-\sin\Omega\cos\omega\cos i\\\sin\Omega\sin\omega+\cos\Omega\cos\omega\cos i\\\cos\omega\sin i\end{pmatrix}$$

$$\frac{1}{a}=\frac{2}{r}-\frac{v^2}{\mu}$$

$$\begin{cases}e\cos E=1-\dfrac{r}{a}\\e\sin E=r\dot{r}/\sqrt{\mu a}\end{cases}$$

$$M=E-e\sin E$$

$$\begin{cases}P_z=\sin i\sin\omega,\quad Q_z=\sin i\cos\omega\\\begin{pmatrix}R_x\\-R_y\\R_z\end{pmatrix}=\begin{pmatrix}\sin i\sin\Omega\\\sin i\cos\Omega\\\cos i\end{pmatrix}\end{cases}$$

$$\begin{cases}\omega=\tan^{-1}(P_z/Q_z)\\\Omega=\tan^{-1}(R_x/(-R_y))\\i=\cos^{-1}R_z\end{cases}$$

开普勒根数与位置速度算例

16 Sep 2019 01:01:39

Satellite-JASON-2_33105: J2000 Position & Velocity

Time (UTCG)	x (km)	y (km)	z (km)	vx (km/sec)	vy (km/sec)	vz (km/sec)
16 Sep 2019 04:00:00.000	-5291.777394	-845.038485	-5558.116835	-3.472599	-4.820868	4.034093
16 Sep 2019 04:01:00.000	-5491.791680	-1132.825631	-5307.530043	-3.192813	-4.769552	4.316635
16 Sep 2019 04:02:00.000	-5674.714794	-1417.087258	-5040.388879	-2.903040	-4.703381	4.585762
16 Sep 2019 04:03:00.000	-5839.974295	-1696.937917	-4757.523749	-2.604178	-4.622557	4.840631
16 Sep 2019 04:04:00.000	-5987.052414	-1971.505544	-4459.814330	-2.297151	-4.527326	5.080443
16 Sep 2019 04:05:00.000	-6115.487742	-2239.934180	-4148.186898	-1.982912	-4.417977	5.304444
16 Sep 2019 04:06:00.000	-6224.876737	-2501.386655	-3823.611511	-1.662437	-4.294847	5.511930
16 Sep 2019 04:07:00.000	-6314.875062	-2755.047226	-3487.099033	-1.336723	-4.158314	5.702244

16 Sep 2019 01:03:43

Satellite-JASON-2_33105: J2000 Classical Orbit Elements

Time (UTCG)	Semi-major Axis (km)	Eccentricity	Inclination (deg)	RAAN (deg)	Arg of Perigee (deg)	True Anomaly (deg)	Mean Anomaly (deg)
16 Sep 2019 04:00:00.000	7712.709754	0.001157	65.972	216.614	153.922	154.061	154.003
16 Sep 2019 04:01:00.000	7713.491720	0.001079	65.974	216.612	157.619	153.563	153.508
16 Sep 2019 04:02:00.000	7714.285891	0.000997	65.975	216.611	161.251	153.129	153.077
16 Sep 2019 04:03:00.000	7715.082398	0.000911	65.976	216.609	164.781	152.797	152.749
16 Sep 2019 04:04:00.000	7715.871334	0.000822	65.978	216.608	168.157	152.619	152.576
16 Sep 2019 04:05:00.000	7716.642885	0.000731	65.979	216.607	171.296	152.678	152.640

第一类贝塞尔函数

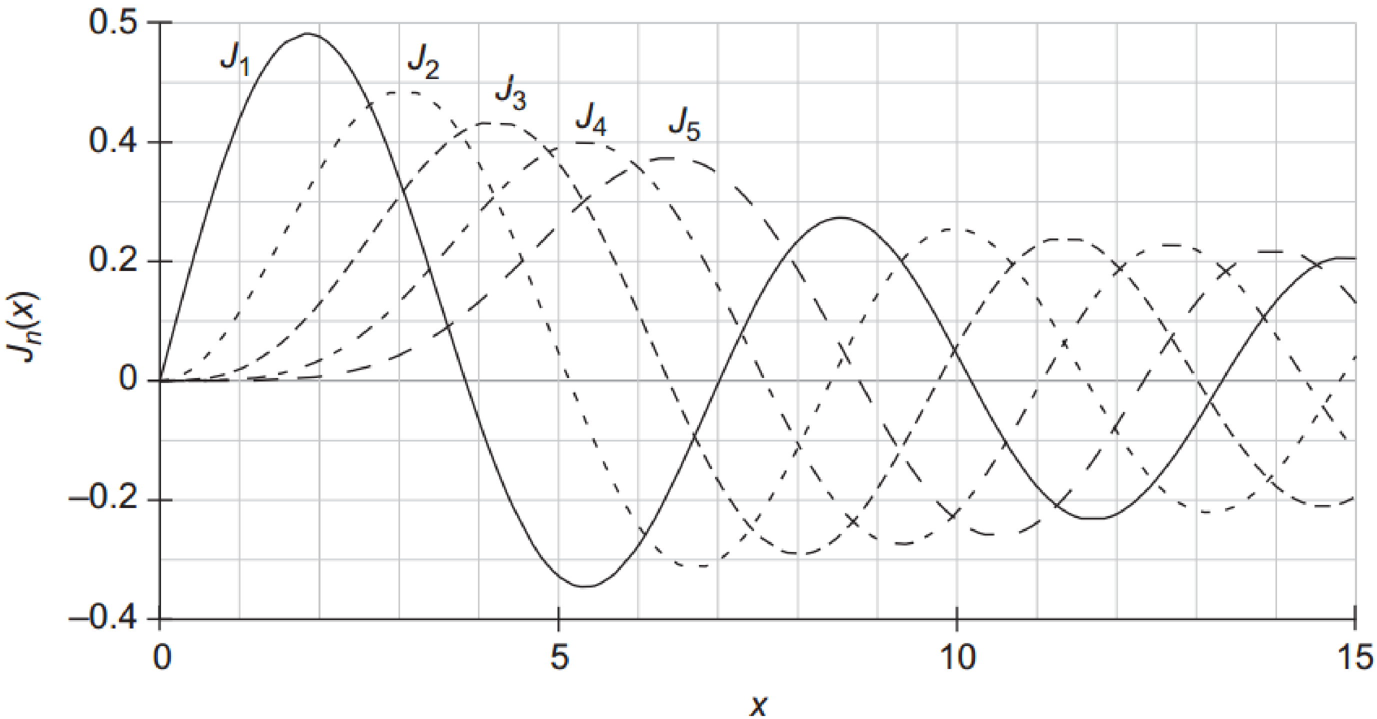
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$

where $J_k(x)$ is the Bessel function

$$J_k(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(kt - x\sin t)\,\mathrm{d}t$$

Using the property of the Bessel functions

$$J_k(x) = \frac{x}{2k}\left[J_{k-1}(x) + J_{k+1}(x)\right]$$



偏近点角的级数展开

$$E = M + \sum_{k=1}^{\infty} \frac{2J_k(ke)}{k} \sin kM.$$

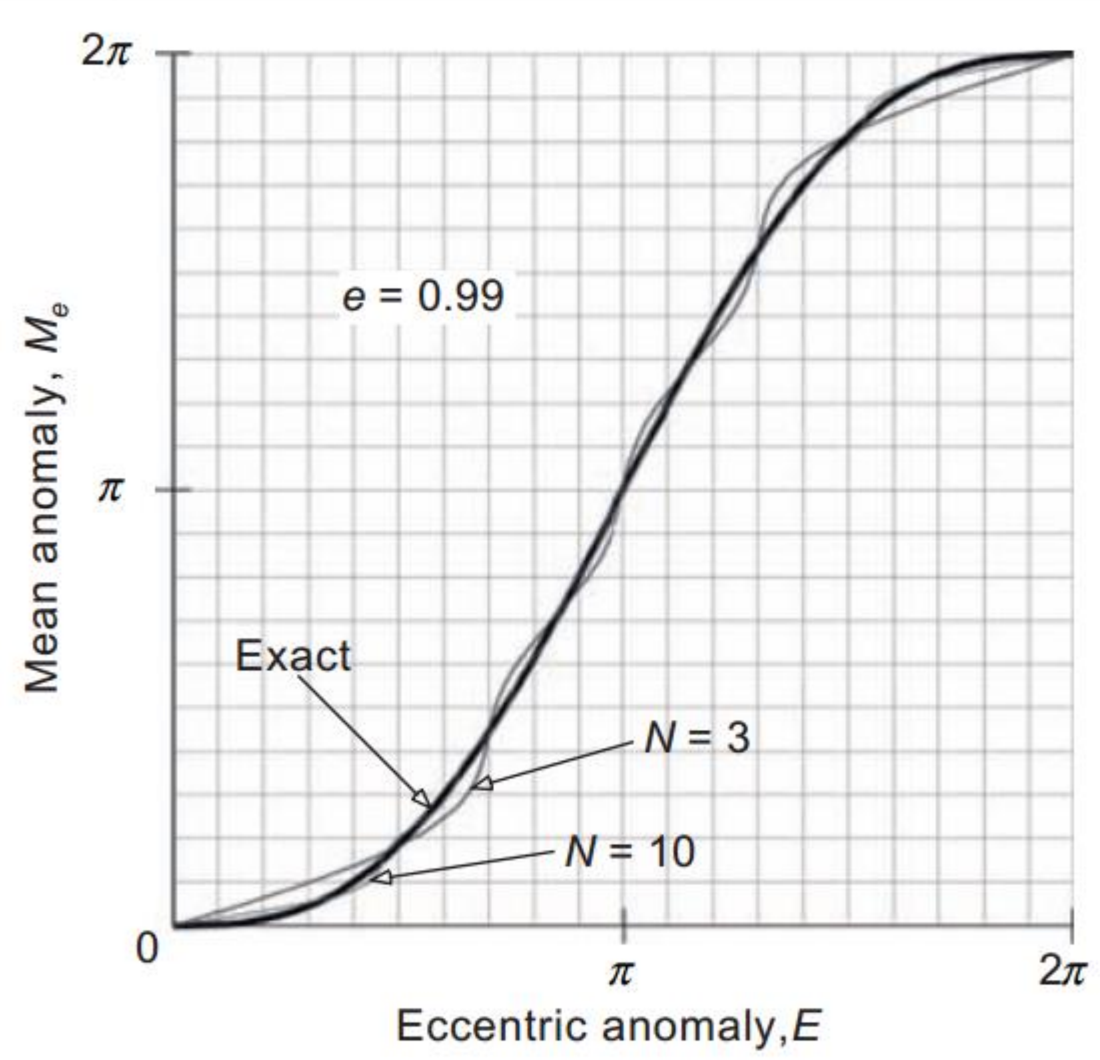
$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{4}\left(\frac{x}{2}\right)^4 - \cdots + \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} + \cdots,$$

$$J_k(x) = \left(\frac{x}{2}\right)^k \frac{1}{k!}\left[1 - \frac{1}{k+1}\left(\frac{x}{2}\right)^2 + \cdots + \frac{(-1)^n}{n!(k+1)(k+2)\cdots(k+n)}\left(\frac{x}{2}\right)^{2n} + \cdots\right].$$

Neglecting terms higher than e^3 we have the approximation

$$\begin{aligned} E &= M + \left(e - \frac{e^3}{8}\right)\sin M + \frac{e^2}{2}\sin 2M + \frac{3e^3}{8}\sin 3M \\ &= M + e\sin M + \frac{e^2}{2}\sin 2M + \frac{e^3}{8}(-\sin M + 3\sin 3M). \end{aligned}$$

贝塞尔函数逼近性能



2π
π
0
e = 0.99
Mean anomaly, M_e
Exact
N = 3
N = 10
π
2π
Eccentric anomaly, E

Sin(nE)与cos(nE)展开

$$\cos nE = -\frac{e}{2}\delta_{n1} + \sum_{k=1}^{\infty} \frac{n}{k}[J_{k-n}(ke) - J_{k+n}(ke)]\cos kM.$$

$$\sin nE = \sum_{k=1}^{\infty} \frac{n}{k}[J_{k-n}(ke) + J_{k+n}(ke)]\sin kM$$

$$\begin{aligned}
\cos E &= -\frac{e}{2} + \left(1 - \frac{3e^2}{8}\right)\cos M + \left(\frac{e}{2} - \frac{e^3}{2}\right)\cos 2M \\
&\quad + \frac{3e^2}{8}\cos 3M + \frac{e^3}{3}\cos 4M, \\
\cos 2E &= \left(-e + \frac{e^3}{12}\right)\cos M + (1 - e^2)\cos 2M \\
&\quad + \left(e - \frac{9e^3}{8}\right)\cos 3M + e^2\cos 4M + \frac{25e^3}{24}\cos 5M, \\
\sin E &= \left(1 - \frac{e^2}{8}\right)\sin M + \left(\frac{e}{2} - \frac{e^3}{8}\right)\sin 2M \\
&\quad + \frac{3e^2}{8}\sin 3M + \frac{e^3}{3}\sin 4M, \\
\sin 2E &= \left(-e + \frac{e^3}{6}\right)\sin M + (1 - e^2)\sin 2M \\
&\quad + \left(e - \frac{9e^3}{8}\right)\sin 3M + e^2\sin 4M + \frac{25e^3}{24}\sin 5M.
\end{aligned}$$

距离作为平近点角的展开

$$\frac{r}{a} = 1 - e\cos E$$
$$= 1 + \frac{e^2}{2} - \sum_{k=1}^{\infty} \frac{e}{k}(J_{k-1}(ke) - J_{k+1}(ke))\cos kM$$
$$= 1 + \frac{e^2}{2} - \sum_{k=1}^{\infty} \frac{2e}{k^2}\frac{\mathrm{d}J_k(ke)}{\mathrm{d}e}\cos kM.$$

Keeping only the terms up to e^3 we get

$$\frac{r}{a} = 1 + \frac{e^2}{2} + \left(-e + \frac{3e^3}{8}\right)\cos M - \frac{e^2}{2}\cos 2M - \frac{3e^3}{8}\cos 3M.$$

$$\frac{a}{r} = \frac{1}{1 - e\cos E} = \frac{1}{\mathrm{d}M/\mathrm{d}E} = \frac{\mathrm{d}E}{\mathrm{d}M}$$
$$= 1 + \sum_{k=1}^{\infty} 2J_k(ke)\cos kM.$$

$$\frac{a}{r} = 1 + \left(e - \frac{e^3}{8}\right)\cos M + e^2\cos 2M + \frac{9e^3}{8}\cos 3M.$$

超几何方程与超几何函数

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

方程系数的每个奇点都是方程的奇点。系数都是解析函数的其余一切值，称为方程的常点。方程三个奇点0,1,∞

$$F(a,b,c;x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!},$$

$$y(x) = AF(a,b,c;x) + Bx^{1-c}F(a-c+1, b-c+1, 2-c; x)$$

超几何函数特例

α	β	γ	z	F
$-n$	β	γ	x	$\sum_{k=0}^{n} \frac{(-n)_k(\beta)_k}{(\gamma)_k} \frac{x^k}{k!}$, where $n = 1, 2, \ldots$
$-n$	β	$-n-m$	x	$\sum_{k=0}^{n} \frac{(-n)_k(\beta)_k}{(-n-m)_k} \frac{x^k}{k!}$, where $n = 1, 2, \ldots$
α	β	β	x	$(1-x)^{-\alpha}$
α	$\alpha+1$	$\frac{1}{2}\alpha$	x	$(1+x)(1-x)^{-\alpha-1}$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	x	$\left(\frac{1+\sqrt{1-x}}{2}\right)^{-2\alpha}$
α	$\alpha+\frac{1}{2}$	2α	x	$\frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	x^2	$\frac{(1+x)^{1-2\alpha}-(1-x)^{1-2\alpha}}{2x(1-2\alpha)}$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	$-\tan^2 x$	$\cos^{2\alpha} x \cos(2\alpha x)$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	x^2	$\frac{1}{2}\left[(1+x)^{-2\alpha}+(1-x)^{-2\alpha}\right]$
α	[illegible]	$2\alpha-1$	x	$2^{2\alpha-2}(1+\sqrt{1-x})^{2-2\alpha}$

勒让德方程解的超几何级数表示

$$(1-z^2)y''_{zz} - 2zy'_z + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]y = 0,$$

For $|1-z| < 2$, the formulas

$$P^\mu_\nu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu,\ 1+\nu,\ 1-\mu;\ \frac{1-z}{2}\right),$$

$$Q^\mu_\nu(z) = A\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} F\left(-\nu,\ 1+\nu,\ 1+\mu;\ \frac{1-z}{2}\right) + B\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu,\ 1+\nu,\ 1-\mu;\ \frac{1-z}{2}\right)$$

$$A = e^{i\mu\pi}\frac{\Gamma(-\mu)\,\Gamma(1+\nu+\mu)}{2\,\Gamma(1+\nu-\mu)}, \quad B = e^{i\mu\pi}\frac{\Gamma(\mu)}{2}, \quad i^2 = -1,$$

For $|z| > 1$,

$$P^\mu_\nu(z) = \frac{2^{-\nu-1}\Gamma(-\frac{1}{2}-\nu)}{\sqrt{\pi}\,\Gamma(-\nu-\mu)} z^{-\nu+\mu-1}(z^2-1)^{-\mu/2} F\left(\frac{1+\nu-\mu}{2},\ \frac{2+\nu-\mu}{2},\ \frac{2\nu+3}{2};\ \frac{1}{z^2}\right)$$
$$+ \frac{2^\nu\Gamma(\frac{1}{2}+\nu)}{\Gamma(1+\nu-\mu)} z^{\nu+\mu}(z^2-1)^{-\mu/2} F\left(-\frac{\nu+\mu}{2},\ \frac{1-\nu-\mu}{2},\ \frac{1-2\nu}{2};\ \frac{1}{z^2}\right),$$

$$Q^\mu_\nu(z) = e^{i\pi\mu}\frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\nu-\mu-1}(z^2-1)^{\mu/2} F\left(\frac{2+\nu+\mu}{2},\ \frac{1+\nu+\mu}{2},\ \frac{2\nu+3}{2};\ \frac{1}{z^2}\right)$$

超几何函数积分表示与部分性质

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\, dt,$$

$$F(\alpha,\beta,\gamma;x) = F(\beta,\alpha,\gamma;x),$$

$$F(\alpha,\beta,\gamma;x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x),$$

$$F(\alpha,\beta,\gamma;x) = (1-x)^{-\alpha} F\left(\alpha,\ \gamma-\beta,\ \gamma;\ \frac{x}{x-1}\right),$$

$$F(\alpha,\beta,\gamma;x) = (1-x)^{-\beta} F\left(\beta,\ \gamma-\alpha,\ \gamma;\ \frac{x}{x-1}\right).$$

(r/a)^n*exp(i*mf)的展开

$$\left(\frac{r}{a}\right)^n \exp(imf) = \sum_{p=-\infty}^{p} X_p^{n,m}(e)\exp(ipM)$$

$$X_p^{n,m}(e) = \int_0^{2\pi} \left(\frac{r}{a}\right)^n \cos(mf - pM)\, dM$$

汉森系数

$$X_p^{n,m}(e) = (1+\beta)^{-(n+1)} \sum_{q=-\infty}^{+\infty} J_q(pe) X_{p,q}^{n,m}$$

$$\beta = \frac{1}{e}\left(1-\sqrt{1-e^2}\right) = \frac{e}{1+\sqrt{1-e^2}}$$

$$X_{p,q}^{n,m} = \begin{cases} (-\beta)^{(p-m)-q} \binom{n-m+1}{p-m-q} F(p-q-n-1, \\ \quad -m-n-1, p-m-q+1, \beta^2) \quad (q \leqslant p-m) \\ (-\beta)^{q-(p-m)} \binom{n+m+1}{q-p+m} F(q-p-n-1, \\ \quad m-n-1, q-p+m+1, \beta^2) \quad (q \geqslant p-m) \end{cases}$$

无奇点根数、正则共轭根数

小偏心率问题、小倾角根数和小偏心率小倾角根数

$$a\ ,\ e_x = e\cos\omega\ ,\ e_y = e\sin\omega\ ,\ i\ ,\ \Omega\ ,\ \tilde{\alpha} = \omega + M$$

$$a\ ,\ e\ ,\ \tilde{\omega} = \omega + \Omega\ ,\ h_x = 2\sin\frac{i}{2}\cos\Omega\ ,\ h_y = 2\sin\frac{i}{2}\sin\Omega\ ,\ M$$

$$a\ ,\ e\cos\tilde{\omega},\ e\sin\tilde{\omega}\ ,2\sin\frac{i}{2}\cos\Omega\ ,2\sin\frac{i}{2}\sin\Omega\ ,\ \lambda \qquad \tilde{\omega} = \omega + \Omega \qquad \tilde{\lambda} = \omega + \Omega + M$$

Delaunay 根数

$$\begin{cases} L = \sqrt{\mu a}, l = M \\ G = L\sqrt{1-e^2}, g = \omega \\ H = G\cos i, h = \Omega \end{cases} \qquad F = -K = \frac{\mu^2}{2L^2} + R \qquad \begin{cases} \dot{p}_i = \dfrac{\partial F}{\partial q_i} \\ \dot{q}_i = -\dfrac{\partial F}{\partial p_i} \end{cases}$$

两行根数

```
AAAAAAAAAAAAAAAAAAAAAAAA
1 NNNNNU NNNNNAAA NNNNN.NNNNNNNN +.NNNNNNNN +NNNNN-N +NNNNN-N N NNNNN
2 NNNNN NNN.NNNN NNN.NNNN NNNNNNN NNN.NNNN NNN.NNNN NN.NNNNNNNNNNNNNN
```

Field	*Column*	*Description*
1.1	01	Line Number of Element Data
1.2	03-07	Satellite Number
1.3	08	Classification
1.4	10-11	International Designator (Last two digits of launch year)
1.5	12-14	International Designator (Launch number of the year)
1.6	15-17	International Designator (Piece of the launch)
1.7	19-20	Epoch Year (Last two digits of year)
1.8	21-32	Epoch (Day of the year and fractional portion of the day)
1.9	34-43	First Time Derivative of the Mean Motion
1.10	45-52	Second Time Derivative of Mean Motion (decimal point assumed)
1.11	54-61	BSTAR drag term (decimal point assumed)
1.12	63	Ephemeris type
1.13	65-68	Element number
1.14	69	Checksum (Modulo 10) (Letters, blanks, periods, plus signs = 0; minus signs = 1)

Field	*Column*	*Description*
2.1	01	Line Number of Element Data
2.2	03-07	Satellite Number
2.3	09-16	Inclination [Degrees]
2.4	18-25	Right Ascension of the Ascending Node [Degrees]
2.5	27-33	Eccentricity (decimal point assumed)
2.6	35-42	Argument of Perigee [Degrees]
2.7	44-51	Mean Anomaly [Degrees]
2.8	53-63	Mean Motion [Revs per day]
2.9	64-68	Revolution number at epoch [Revs]
2.10	69	Checksum (Modulo 10)

两行根数算例

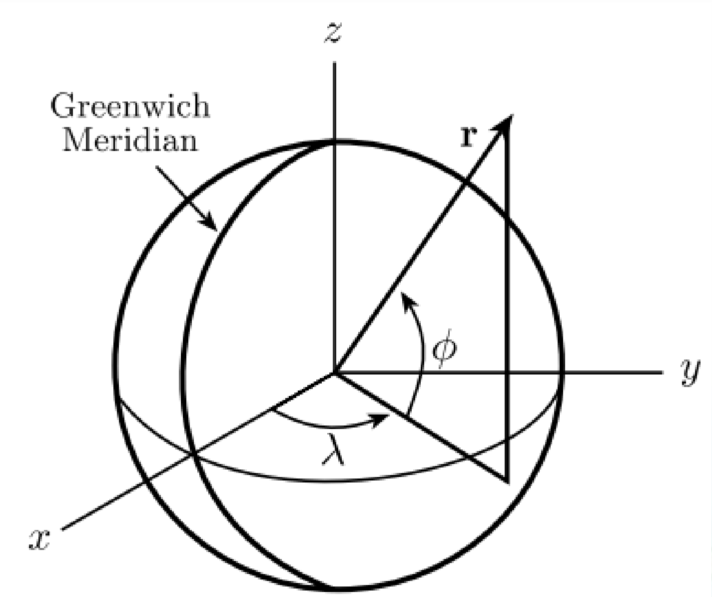
```
                                                          16 Sep 2019 01:06:42
Satellite-JASON-2_33105


                     Two Line Element Set
----------------------------------------------------------------------
1 33105U 08032A   17088.90414795 -.00000066  00000-0 -20983-4 0  9998
2 33105  66.0401 286.3042 0007614 274.4658 183.4887 12.80932272410381
```

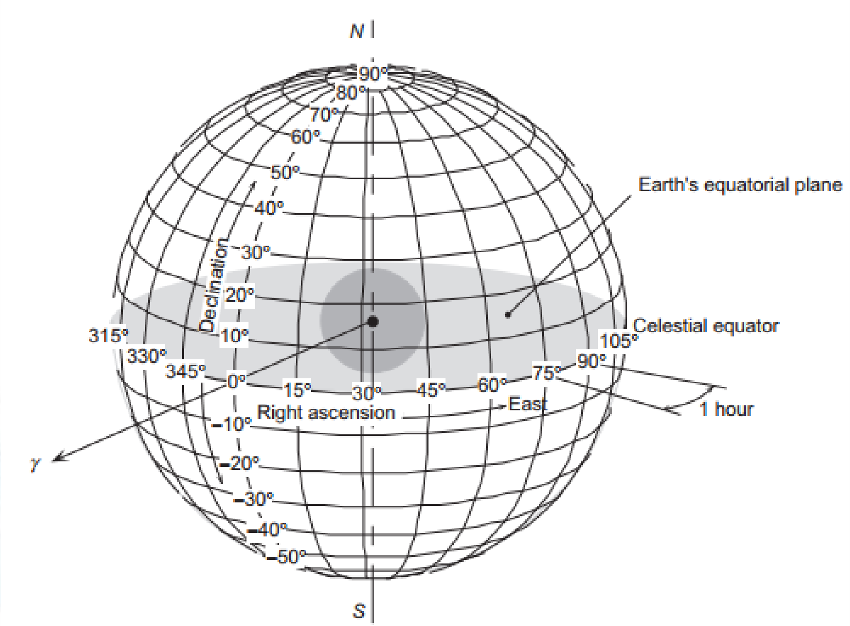
16 Sep 2019 01:01:39

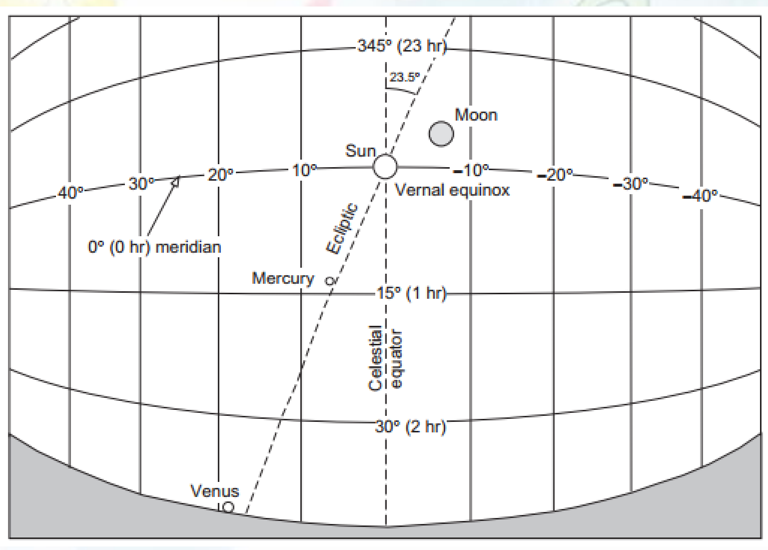
Satellite-JASON-2_33105: J2000 Position & Velocity

Time (UTCG)	x (km)	y (km)	z (km)	vx (km/sec)	vy (km/sec)	vz (km/sec)
16 Sep 2019 04:00:00.000	-5291.777394	-845.038485	-5558.116835	-3.472599	-4.820868	4.034093
16 Sep 2019 04:01:00.000	-5491.791680	-1132.825631	-5307.530043	-3.192813	-4.769552	4.316635
16 Sep 2019 04:02:00.000	-5674.714794	-1417.087258	-5040.388879	-2.903040	-4.703381	4.585762
16 Sep 2019 04:03:00.000	-5839.974295	-1696.937917	-4757.523749	-2.604178	-4.622557	4.840631
16 Sep 2019 04:04:00.000	-5987.052414	-1971.505544	-4459.814330	-2.297151	-4.527326	5.080443
16 Sep 2019 04:05:00.000	-6115.487742	-2239.934180	-4148.186898	-1.982912	-4.417977	5.304444
16 Sep 2019 04:06:00.000	-6224.876737	-2501.386655	-3823.611511	-1.662437	-4.294847	5.511930
16 Sep 2019 04:07:00.000	-6314.875062	-2755.047226	-3487.099033	-1.336723	-4.158314	5.702244

星下点轨迹

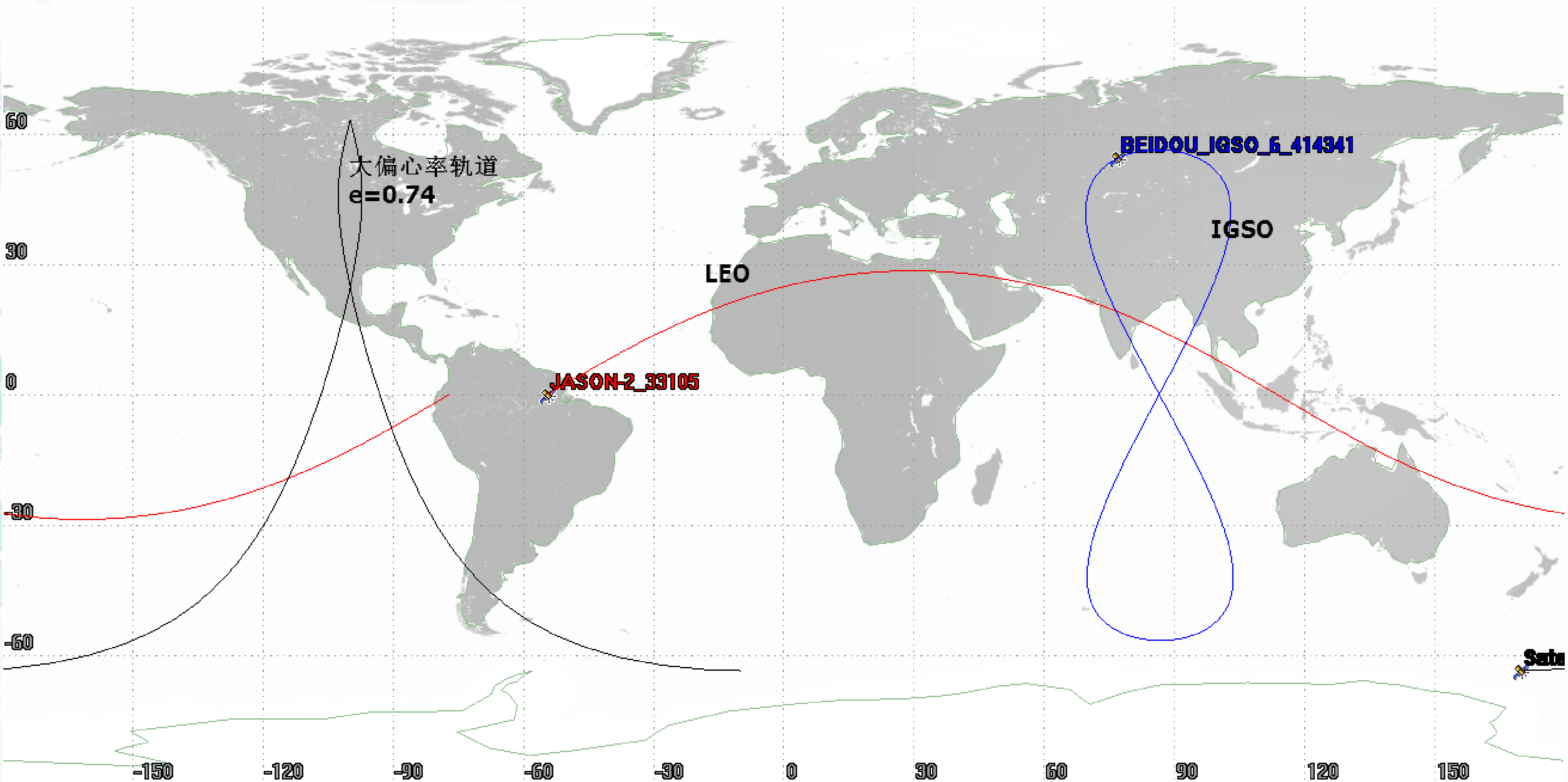






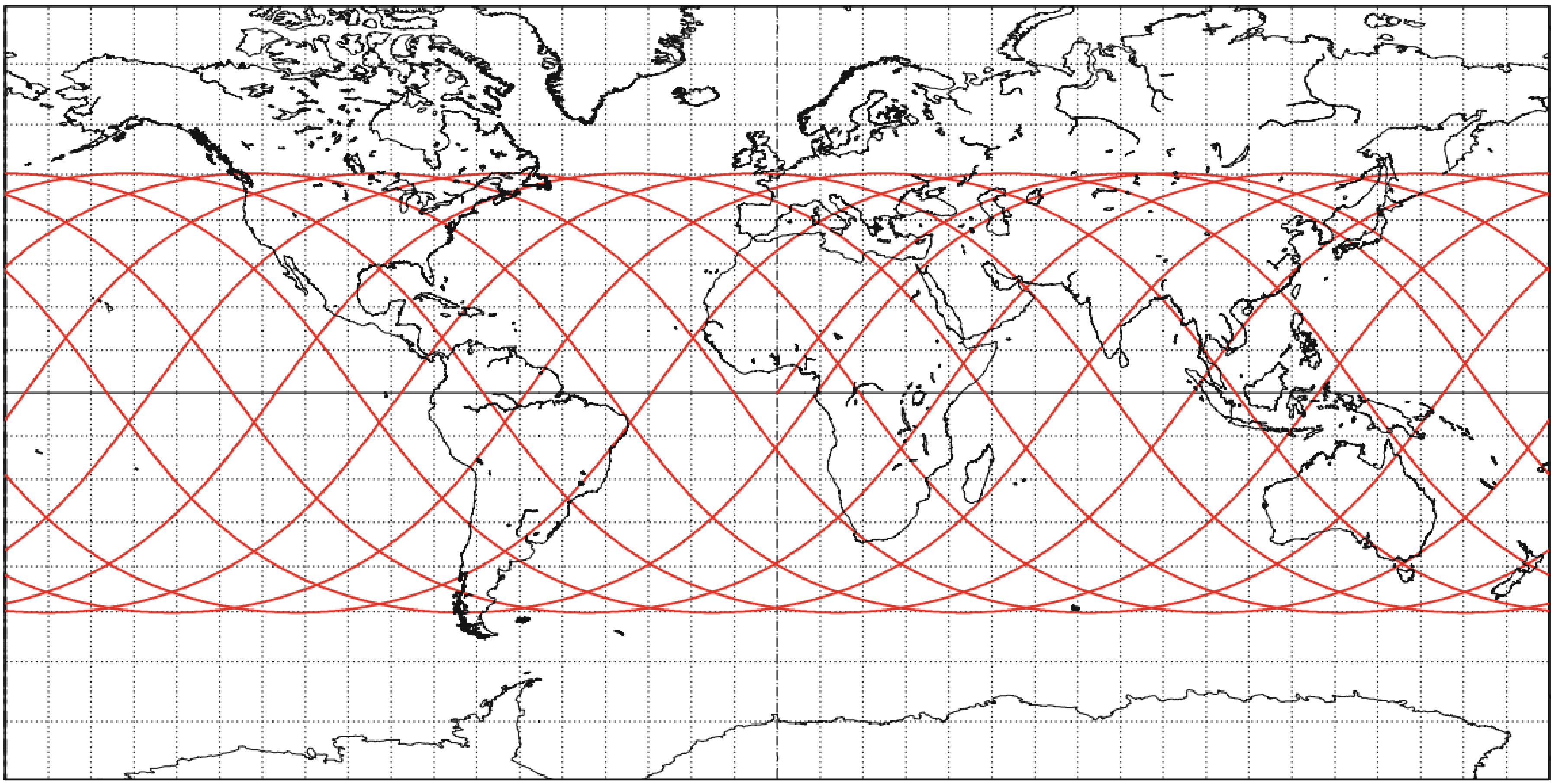
卫星星下点轨迹是星下点在地球表面通过的路径。星下点地理纬度为卫星赤纬，地理经度为卫星赤经与t时刻格林尼治恒星时之差.（satellite orbit）

IGSO、LEO与大偏心率轨道星下点

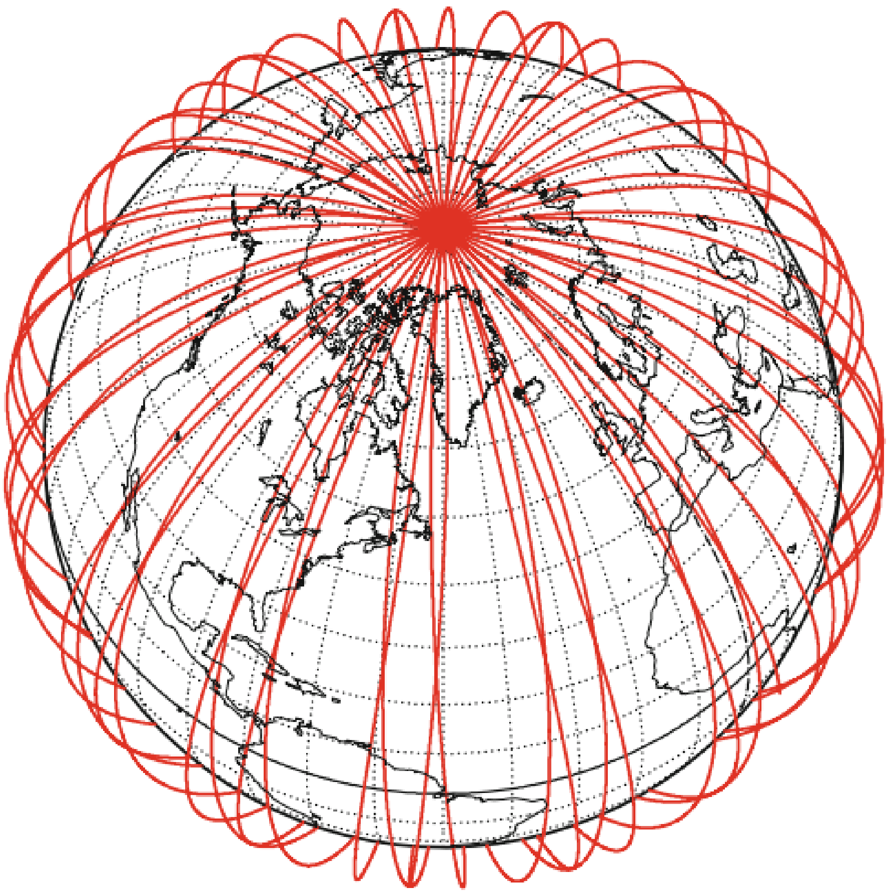


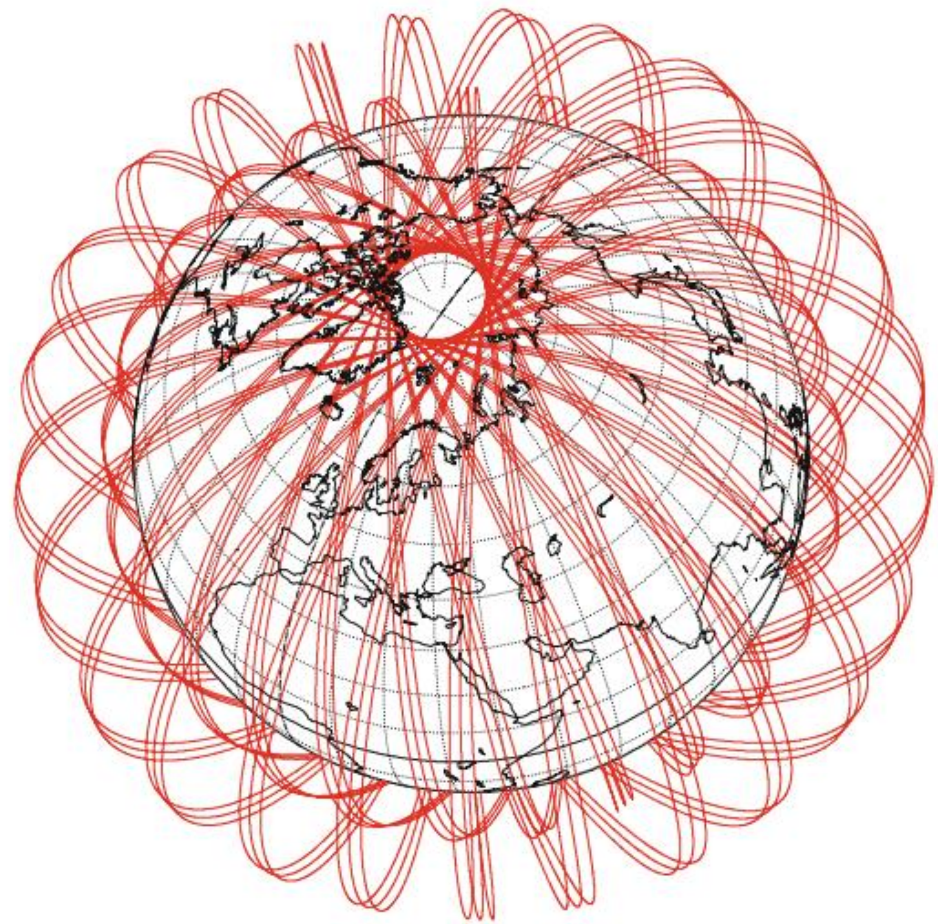
多圈星下点

Altitude=1488.5 km,
Period = 115.65 min
Inclination=50.01deg

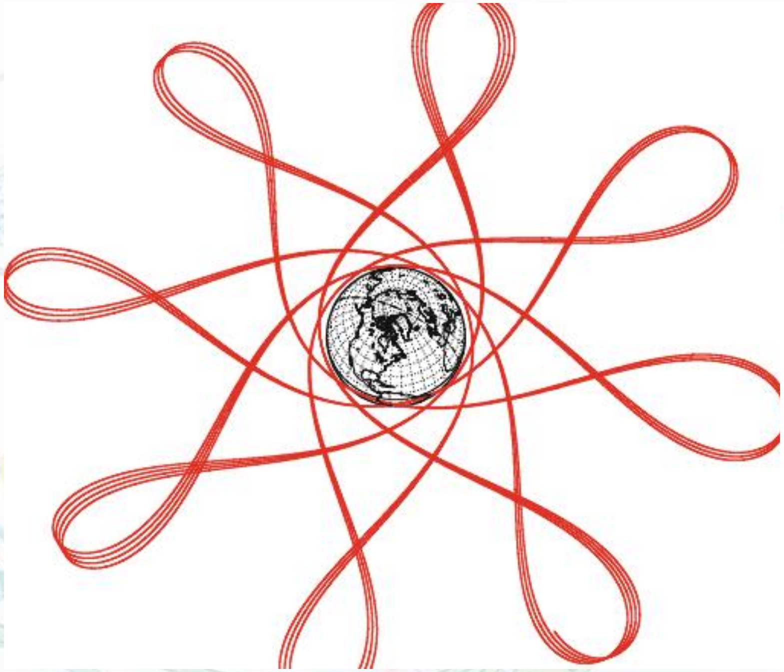


空间视角





大偏心率轨道

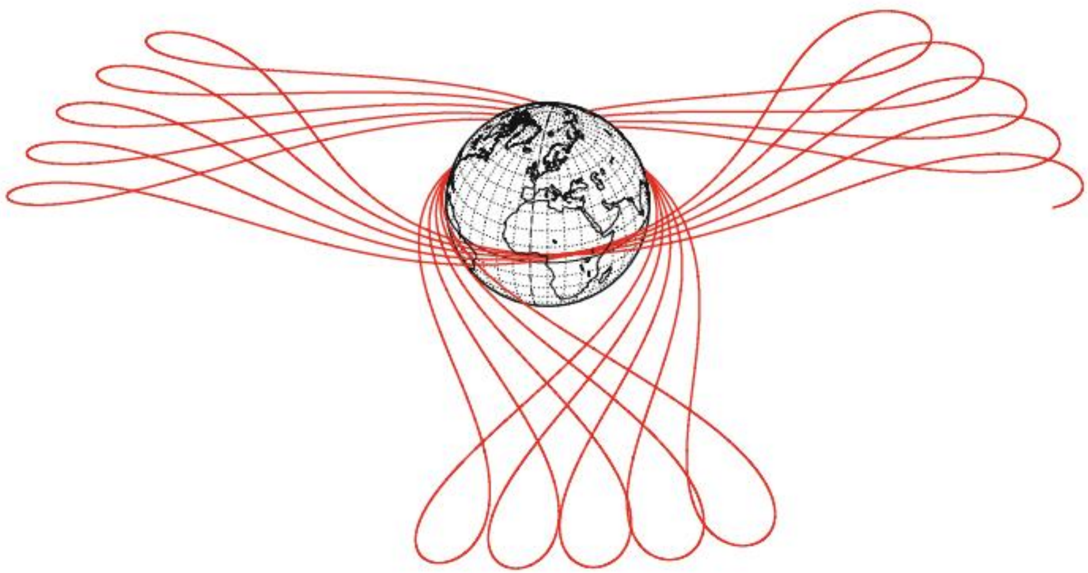


Equiv. altit. = 15559.4 km a =21937.541 km

Inclination = 9.95 ° e = 0.682033

Period = 538.26 min * rev/day = 2.68

h_a = 30522 km; h_p = 597 km; arg. perigee: +24.96 °



Equiv. altit. = 14233.5 km a =20611.604 km

Inclination = 7.15 ° e = 0.679397

Period = 490.12 min * Revol./d.= 2.94

h_a = 28237 km ; h_p = 230 km ; arg. perigee: +55.73 °


Q&A!