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第十二讲 变分方程

- 微分动力系统关于初值与参数的偏导数
- 二体A矩阵的计算
- 其他摄动力A矩阵的计算
- 状态方程与变分方程联合数值积分

状态量关于初值偏导数

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} \dot{\mathbf{r}}(t) \\ \ddot{\mathbf{r}}(t) \end{pmatrix}$$

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(t, \mathbf{x}(t), \mathbf{p}) \\ \mathbf{x}(t) & = & \mathbf{x}_0 + \int_0^t \mathbf{f}(\tau, \mathbf{x}(\tau), \mathbf{p})\, d\tau \end{cases}$$

$$\begin{cases} \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(t) & = & \mathbf{I} + \int_0^t \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\tau) \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\tau) d\tau \\ \mathbf{\Phi}(t) & = & \mathbf{\Phi}(0) + \int_0^t \mathbf{F}(\tau) \mathbf{\Phi}(\tau) d\tau \\ \dot{\mathbf{\Phi}}(t) & = & \mathbf{F}(t)\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I} \end{cases}$$

$$\mathbf{\Phi}(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(t) \text{ and } \mathbf{F}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t).$$

状态量关于参数偏导数

$$\begin{cases} \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(t) & = & \int_0^t \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\tau) \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(\tau) + \frac{\partial \mathbf{f}}{\partial \mathbf{p}}(\tau) \right] d\tau \\ \mathbf{\Psi}(t) & = & \int_0^t \left[\mathbf{F}(\tau)\mathbf{\Psi}(\tau) + \mathbf{G}(\tau) \right] d\tau \\ \dot{\mathbf{\Psi}}(t) & = & \mathbf{F}(t)\mathbf{\Psi}(t) + \mathbf{G}(t), \quad \mathbf{\Psi}(0) = \mathbf{0} \end{cases}$$

$\mathbf{\Psi}(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}(t)$ and $\mathbf{G}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{p}}(t)$.

二体的F矩阵

$$\ddot{\mathbf{r}} = -\frac{GM_\oplus}{r^3}\mathbf{r}$$

$$\frac{\partial r^n}{\partial \mathbf{r}} = \frac{\partial (x^2+y^2+z^2)^{n/2}}{\partial \mathbf{r}} = n \cdot r^{n-2} \cdot \mathbf{r}^T$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} = \frac{GM_\oplus}{r^5}\begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix}$$

非球形引力场F矩阵

$$U = \frac{GM}{r}\left[1 + \sum_{n=2}^{nmax}\sum_{m=0}^{n}\left[\frac{a_e}{r}\right]^n P_m^n(\sin\phi')[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\right]$$

变分方程需要计算矩阵 $$[U_{2c}]_{i,j} = \frac{\partial^2 U}{\partial r_i \partial r_j}$$

$$U_{2c} = C_1^T U_2 C_1 + \sum_{k=1}^{3}\frac{\partial U}{\partial e_k} C_{2k}$$

e_k ranges over the elements r

U_2 is the matrix whose i, j^{th} element is given by $\partial^2 U \partial e_i \partial e_j$

C_1 is the matrix whose i, j^{th} element is given by $\frac{\partial e_i}{\partial r_j}$

C_{2k} is the matrix whose i, j^{th} element is given by $\frac{\partial^2 e_k}{\partial r_i \partial r_j}$

非球形引力场F矩阵

$$\frac{\partial^2 U}{\partial r^2} = \frac{2GM}{r^3} + \frac{GM}{r^3} \sum_{n=2}^{nmax} (n+1)(n+2) \left[\frac{a_e}{r}\right]^n$$

$$\times \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_n^m(\sin\phi')$$

$$\frac{\partial^2 U}{\partial r \partial \phi'} = \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda$$

$$+ S_{nm} \sin m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin\phi')$$

$$\frac{\partial^2 U}{\partial r \partial \lambda} = \frac{GM}{r^2} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} m(-C_{nm} \sin m\lambda$$

$$+ S_{nm} \cos m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin\phi')$$

非球形引力场F矩阵

$$\frac{\partial^2 U}{\partial \phi'^2} = \frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \frac{\partial^2}{\partial \phi'^2} [P_n^m(\sin \phi')]$$

$$\frac{\partial^2 U}{\partial \phi' \partial \lambda} = \frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} m(-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda) \frac{\partial}{\partial \phi'} P_n^m(\sin \phi')$$

$$\frac{\partial^2 U}{\partial \lambda^2} = -\frac{GM}{r} \sum_{n=2}^{nmax} (n+1) \left[\frac{a_e}{r}\right]^n \sum_{m=0}^{n} m^2(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_n^m(\sin \phi')$$

非球形引力场F矩阵

$$\frac{\partial}{\partial\phi'}[P_n^m(\sin\phi')] = P_n^{m+1}(\sin\phi') - m\tan\phi' P_n^m(\sin\phi')$$

$$\frac{\partial^2}{\partial\phi'}[P_n^m(\sin\phi')] = P_n^{m+2}(\sin\phi') - (m+1)\tan\phi' P_n^{m+1}(\sin\phi') \\ -m\tan\phi'[P_n^{m+1}(\sin\phi') - m\tan\phi' P_n^m(\sin\phi')] \\ -m\sec^2\phi' P_n^m(\sin\phi')$$

$$\frac{\partial U}{\partial \sin\phi'} = \frac{\partial U}{\partial\phi'}\frac{\partial\phi'}{\partial\sin\phi'}$$

$$\frac{\partial^2 U}{\partial \sin\phi'^2} = \frac{\partial\phi'}{\partial\sin\phi'}\left[\frac{\partial^2 U}{\partial\phi'^2}\right]\frac{\partial\phi'}{\partial\sin\phi'} + \frac{\partial U}{\partial\phi'}\frac{\partial^2\phi'}{\partial\sin\phi'^2}$$

$$\frac{\partial\phi'}{\partial\sin\phi} = \sec\phi'$$

$$\frac{\partial^2\partial\phi'}{\partial\sin\phi'^2} = \sin\phi'\sec^3\phi'$$

非球形引力场F矩阵

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r}$$

$$\frac{\partial \sin\phi'}{\partial r_i} = -\frac{z r_i}{r^3} + \frac{1}{r}\frac{\partial x}{\partial r_i}$$

$$\frac{\partial \lambda}{\partial r_i} = \frac{1}{x^2+y^2}\left[x\frac{\partial y}{\partial r_i} - y\frac{\partial x}{\partial r_i}\right]$$

$$\frac{\partial^2 r}{\partial r_i \partial r_j} = \frac{r_i r_j}{r^3} + \frac{1}{r}\frac{\partial r_i}{\partial r_j}$$

$$\frac{\partial^2 \sin\phi'}{\partial r_i \partial r_j} = \frac{3 z r_i r_j}{r^5} - \frac{1}{r^3}\left[r_j\frac{\partial z}{\partial r_i} + r_i\frac{\partial z}{\partial r_j} + z\frac{\partial r_i}{\partial r_j}\right]$$

$$\frac{\partial^2 \lambda}{\partial r_i \partial r_j} = \frac{-2 r_j}{(x^2+y^2)^2}\left[x\frac{\partial y}{\partial r_i} - y\frac{\partial x}{\partial r_i}\right] + \frac{1}{(x^2+y^2)}\left[\frac{\partial x}{\partial r_j}\frac{\partial y}{\partial r_j} - \frac{\partial y}{\partial r_j}\frac{\partial x}{\partial r_j}\right]$$

关于引力场系数的偏导数

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial r}\right]=(n+1)\frac{GM}{r^2}\left[\frac{a_e}{r}\right]^n\cos(m\lambda)P_n^m(\sin\phi')$$

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \lambda}\right]=m\frac{GM}{r}\left[\frac{a_e}{r}\right]^n\sin(m\lambda)P_n^m(\sin\phi')$$

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \phi'}\right]=\frac{-GM}{r}\left[\frac{a_e}{r}\right]^n\cos(m\lambda)\left[P_n^{m+1}(\sin\phi')-m\tan\phi' P_n^m(\sin\phi')\right]$$

$$\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial x}\right]=\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial r}\right]\frac{\partial r}{\partial x}+\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \lambda}\right]\frac{\partial \lambda}{\partial x}+\frac{\partial}{\partial C_{nm}}\left[-\frac{\partial U}{\partial \phi'}\right]\frac{\partial \phi'}{\partial x}$$

第三体引力位

$$R_d = \frac{Gm_d}{r_d}\left[\left[1 - \frac{2r}{r_d}S + \frac{r^2}{r_d^2}\right]^{-\frac{1}{2}} - \frac{r}{r_d}S\right]$$

m_d is the mass of the disturbing body.

$\bar{r}_d$ is the geocentric true of date position vector to the disturbing body.

S is equal to the cosine of the enclosed angle between $\bar{r}$ and $\bar{r_d}$.

$\bar{r}$ is the geocentric true of date position vector of the satellite.

G is the universal gravitational constant.

第三体A矩阵

$$\bar{a_d} = -Gm_d\left[\frac{\bar{d}}{D} + \left[\frac{\bar{r}}{r_d}\right]\right]$$

$$\bar{d} = \bar{r} - \bar{r}_d$$

$$D_d = [r_d^2 - 2rr_dS + r^2]^{\frac{3}{2}}$$

$$\frac{\partial^2 R_d}{\partial r_i \partial r_j} = -\frac{GMm_d}{D_d}\left[\frac{\partial r_i}{\partial r_j} + \frac{3d_i d_j}{D_d^{\frac{2}{3}}}\right]$$

光压F矩阵

$$\frac{\partial \ddot{\boldsymbol{r}}}{\partial \boldsymbol{r}} = +P_{\odot} C_r \frac{A}{m} \mathrm{AU}^2 \left(\frac{1}{|\boldsymbol{r}-\boldsymbol{s}|^3} \mathbf{1}_{3\times 3} - 3(\boldsymbol{r}-\boldsymbol{s}) \frac{(\boldsymbol{r}-\boldsymbol{s})^T}{|\boldsymbol{r}-\boldsymbol{s}|^5} \right)$$

$$\frac{\partial \ddot{\boldsymbol{r}}}{C_r} = \frac{1}{C_r} \ddot{\boldsymbol{r}} = -P_{\odot} \frac{A}{m} \frac{\boldsymbol{r}_{\odot}}{r_{\odot}^3} \mathrm{AU}^2$$

大气阻力F矩阵

$$\ddot{\boldsymbol{r}} = -\frac{1}{2} C_D \frac{A}{m} \rho\, v_r \boldsymbol{v}_r \quad \text{with} \quad \boldsymbol{v}_r = \boldsymbol{v} - \boldsymbol{\omega}_\oplus \times \boldsymbol{r}$$

$$\frac{\partial \ddot{\boldsymbol{r}}}{\partial C_D} = -\frac{1}{2} \frac{A}{m} \rho\, v_r \boldsymbol{v}_r$$

$$\frac{\partial \ddot{\boldsymbol{r}}}{\partial \boldsymbol{v}} = -\frac{1}{2} C_D \frac{A}{m} \rho \left(\frac{\boldsymbol{v}_r \boldsymbol{v}_r^T}{v_r} + v_r \mathbf{1} \right)$$

$$\frac{\partial \ddot{\boldsymbol{r}}}{\partial \boldsymbol{r}} = -\frac{1}{2} C_D \frac{A}{m} v_r \boldsymbol{v}_r \frac{\partial \rho}{\partial \boldsymbol{r}} - \frac{1}{2} C_D \frac{A}{m} \rho \left(\frac{\boldsymbol{v}_r \boldsymbol{v}_r^T}{v_r} + v_r \mathbf{1} \right) \frac{\partial \boldsymbol{v}_r}{\partial \boldsymbol{r}}$$

动力方程与变分方程联合积分（二阶）

$$\begin{cases} \dfrac{d^2}{dt^2}\left(\dfrac{\partial \vec{R}}{\partial \vec{R}_0}\right) = \dfrac{\partial \vec{A}}{\partial \vec{R}}\dfrac{\partial \vec{R}}{\partial \vec{R}_0} + \dfrac{\partial \vec{A}}{\partial \dot{\vec{R}}}\dfrac{d}{dt}\left(\dfrac{\partial \vec{R}}{\partial \vec{R}_0}\right) \\ \dfrac{d^2}{dt^2}\left(\dfrac{\partial \vec{R}}{\partial \dot{\vec{R}}_0}\right) = \dfrac{\partial \vec{A}}{\partial \vec{R}}\dfrac{\partial \vec{R}}{\partial \dot{\vec{R}}_0} + \dfrac{\partial \vec{A}}{\partial \dot{\vec{R}}}\dfrac{d}{dt}\left(\dfrac{\partial \vec{R}}{\partial \dot{\vec{R}}_0}\right) \\ \dfrac{d^2}{dt^2}\left(\dfrac{\partial \vec{R}}{\partial \vec{p}_d}\right) = \dfrac{\partial \vec{A}}{\partial \vec{R}}\dfrac{\partial \vec{R}}{\partial \vec{p}_d} + \dfrac{\partial \vec{A}}{\partial \dot{\vec{R}}}\dfrac{d}{dt^2}\left(\dfrac{\partial \vec{R}}{\partial \vec{p}_d}\right) + \dfrac{\partial \vec{A}}{\partial \vec{p}_d} \end{cases}$$

$$\left(\frac{\partial \vec{R}}{\partial \vec{R}_0}\right)_{t=t_0} = I_{3\times3} \quad \left(\frac{\partial \vec{R}}{\partial \dot{\vec{R}}_0}\right)_{t=t_0} = 0, \quad \left(\frac{\partial \vec{R}}{\partial \vec{p}_d}\right)_{t=t_0} = 0$$

$$\left[\frac{d}{dt}\left(\frac{\partial \vec{R}}{\partial \vec{R}_0}\right)\right]_{t=t_0} = 0, \quad \left[\frac{d}{dt}\left(\frac{\partial \vec{R}}{\partial \vec{p}_d}\right)\right]_{t=t_0} = 0 \quad \left[\frac{d}{dt}\left(\frac{\partial \vec{R}}{\partial \dot{\vec{R}}_0}\right)\right]_{t=t_0} = I_{3\times3}$$

动力方程与变分方程联合积分（二阶）

$$\vec{Z}=\begin{pmatrix}\vec{R}\\ \dfrac{\partial\vec{R}}{\partial x_0}\\ \dfrac{\partial\vec{R}}{\partial y_0}\\ \dfrac{\partial\vec{R}}{\partial z_0}\\ \dfrac{\partial\vec{R}}{\partial\dot{x}_0}\\ \dfrac{\partial\vec{R}}{\partial\dot{y}_0}\\ \dfrac{\partial\vec{R}}{\partial\dot{z}_0}\\ \dfrac{\partial\vec{R}}{\partial p_{d1}}\\ \vdots\\ \dfrac{\partial\vec{R}}{\partial p_{dNDP}}\end{pmatrix}$$

$$\frac{d^2\vec{Z}}{dt^2}=\frac{d^2}{dt^2}\begin{pmatrix}\vec{R}\\ \dfrac{\partial\vec{R}}{\partial x_0}\\ \vdots\\ \dfrac{\partial\vec{R}}{\partial\dot{x}_0}\\ \vdots\\ \dfrac{\partial\vec{R}}{\partial p_1}\\ \vdots\end{pmatrix}=\begin{pmatrix}\vec{A}\\ \dfrac{\partial\vec{A}}{\partial\vec{R}}\dfrac{\partial\vec{R}}{\partial x_0}+\dfrac{\partial\vec{A}}{\partial\dot{\vec{R}}}\dfrac{d}{dt}\left(\dfrac{\partial\vec{R}}{\partial x_0}\right)\\ \vdots\\ \dfrac{\partial\vec{A}}{\partial\vec{R}}\dfrac{\partial\vec{R}}{\partial\dot{x}_0}+\dfrac{\partial\vec{A}}{\partial\dot{\vec{R}}}\dfrac{d}{dt}\left(\dfrac{\partial\vec{R}}{\partial\dot{x}_0}\right)\\ \vdots\\ \dfrac{\partial\vec{A}}{\partial\vec{R}}\dfrac{\partial\vec{R}}{\partial p_{d1}}+\dfrac{\partial\vec{A}}{\partial\dot{\vec{R}}}\dfrac{d}{dt}\left(\dfrac{\partial\vec{R}}{\partial p_{d1}}\right)+\dfrac{\partial\vec{A}}{\partial p_{d1}}\\ \vdots\end{pmatrix}=\vec{f}$$

线性定常系统状态转移矩阵范例：

对于动力学系统

$$\dot{\boldsymbol{X}} = \begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

初始条件为

$$x_1(t_0) = x_{10}, x_2(t_0) = x_{20}$$

$$\boldsymbol{\Phi}(t, t_0) \equiv \frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{X}(t_0)}$$

?

求解方法1

$$\frac{dx_1}{dt} = x_1$$

$$x_1(t) = ce^{t}, t = t_0, x_1 = x_{10}$$

$$c = x_{10}e^{-t_0}$$

$$x_1 = x_{10}e^{t-t_0}$$

$$dx_2 = \alpha x_{10}e^{t-t_0}dt$$

$$x_2 = \alpha x_{10}e^{t-t_0} + c_2, t = t_0, x_2 = x_{20}$$

$$x_2(t) = b + \alpha x_{10}\left(e^{t-t_0} - 1\right)$$

$$\boldsymbol{\Phi}(t,t_0) \equiv \frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{X}(t_0)} = \begin{bmatrix} \dfrac{\partial x_1(t)}{\partial x_1(t_0)} & \dfrac{\partial x_1(t)}{\partial x_2(t_0)} \\ \dfrac{\partial x_2(t)}{\partial x_1(t_0)} & \dfrac{\partial x_2(t)}{\partial x_2(t_0)} \end{bmatrix} = \begin{bmatrix} e^{t-t_0} & 0 \\ \alpha\left(e^{t-t_0} - 1\right) & 1 \end{bmatrix}$$

求解方法2：

$$\begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad \Phi(t_0, t_0) = I$$

$$\begin{cases} \dot{\phi}_{11} = \phi_{11} \\ \dot{\phi}_{21} = \alpha\phi_{11} \\ \dot{\phi}_{12} = \phi_{12} \\ \dot{\phi}_{22} = \alpha\phi_{12} \end{cases}$$

$$\phi_{11} = ce^{t}, \phi_{11}(t_0) = 1, c = e^{-t_0}, \phi_{11} = e^{(t-t_0)}$$

$$\dot{\phi}_{21} = \alpha e^{t-t_0}, \phi_{21} = \alpha e^{t-t_0} + c, \phi_{21}(t_0) = 0$$

$$c = -\alpha \qquad \phi_{21} = \alpha\left(e^{t-t_0} - 1\right)$$

$$\phi_{12} = 0 \qquad \phi_{22} = 1$$


Q&A!