中国科学院上海天文台

中国科学院大学
University of Chinese Academy of Sciences

空间飞行器精密定轨

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第十四讲 参数估计与计算方法

- 超定方程求解
- 法方程处理
- 正交变换方法
- 滤波方法
- 数值算法实现

线性超定方程

$$
\begin{aligned}
\mathbf{y}_1 &= H_1\mathbf{x}_k + \boldsymbol{\epsilon}_1; \qquad w_1 \\
\mathbf{y}_2 &= H_2\mathbf{x}_k + \boldsymbol{\epsilon}_2; \qquad w_2 \\
&\vdots \quad \vdots \quad \vdots \\
\mathbf{y}_\ell &= H_\ell\mathbf{x}_k + \boldsymbol{\epsilon}_\ell; \qquad w_\ell
\end{aligned}
$$

$$
\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_\ell \end{bmatrix}; \quad H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_\ell \end{bmatrix};
$$

$$
\boldsymbol{\epsilon} = \begin{bmatrix} \boldsymbol{\epsilon}_1 \\ \boldsymbol{\epsilon}_2 \\ \vdots \\ \boldsymbol{\epsilon}_\ell \end{bmatrix}; \quad W = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & w_\ell \end{bmatrix}
$$

最优估值

$$J(\mathbf{x}_k) = 1/2\boldsymbol{\epsilon}^T W \boldsymbol{\epsilon} = \sum_{i=1}^{\ell} 1/2\boldsymbol{\epsilon}_i^T w_i \boldsymbol{\epsilon}_i$$

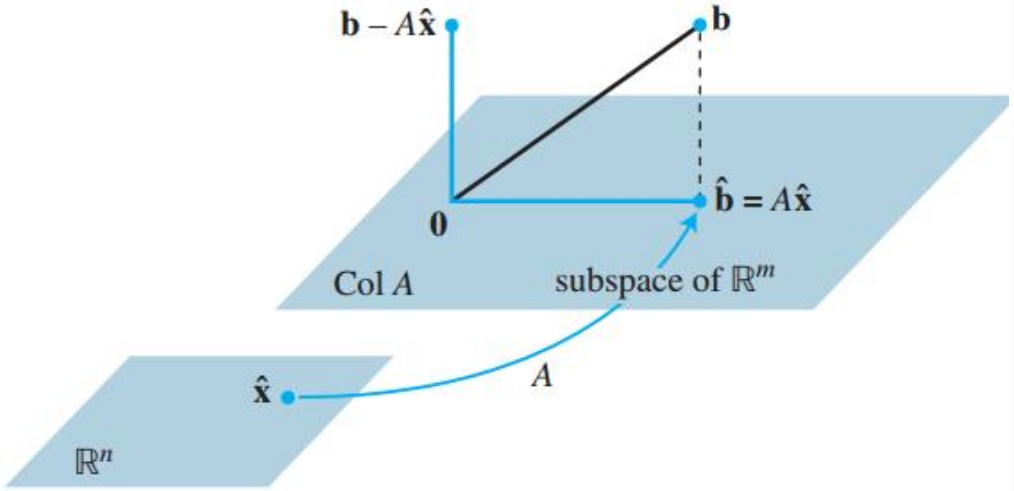
$$J(\mathbf{x}_k) = 1/2(\mathbf{y} - H\mathbf{x}_k)^T W (\mathbf{y} - H\mathbf{x}_k).$$

$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 = -(y - H\mathbf{x}_k)^T W H = -H^T W (\mathbf{y} - H\mathbf{x}_k).$$

$$(H^T W H)\mathbf{x}_k = H^T W \mathbf{y}.$$

$$P_k = (H^T W H)^{-1}.$$

几何解释



The least-squares solution $\hat{\mathbf{x}}$ is in $\mathbb{R}^n$.

Suppose $\hat{\mathbf{x}}$ satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$. By the Orthogonal Decomposition Theorem in Section 6.3, the projection $\hat{\mathbf{b}}$ has the property that $\mathbf{b} - \hat{\mathbf{b}}$ is orthogonal to Col A, so $\mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to each column of A. If $\mathbf{a}_j$ is any column of A, then $\mathbf{a}_j \cdot (\mathbf{b} - A\hat{\mathbf{x}}) = 0$, and $\mathbf{a}_j^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0$. Since each $\mathbf{a}_j^T$ is a row of A^T,

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$$
$$A^T\mathbf{b} - A^TA\hat{\mathbf{x}} = \mathbf{0}$$
$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$$

These calculations show that each least-squares solution of $A\mathbf{x} = \mathbf{b}$ satisfies the equation

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

The matrix equation (3) represents a system of equations called the **normal equations** for $A\mathbf{x} = \mathbf{b}$. A solution of (3) is often denoted by $\hat{\mathbf{x}}$.

先验信息与序贯问题

$$\boldsymbol{y}_1 + \boldsymbol{v}_1 = \boldsymbol{A}_1\, \boldsymbol{p}_1 \qquad \text{with} \qquad \boldsymbol{D}\left(\boldsymbol{y}_1\right) = \sigma_1^2 \boldsymbol{P}_1^{-1}$$

$$\boldsymbol{y}_2 + \boldsymbol{v}_2 = \boldsymbol{A}_2\, \boldsymbol{p}_2 \qquad \text{with} \qquad \boldsymbol{D}\left(\boldsymbol{y}_2\right) = \sigma_2^2 \boldsymbol{P}_2^{-1}$$

$$\begin{bmatrix} \widehat{\boldsymbol{p}}_1 \\ \widehat{\boldsymbol{p}}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_{p_1} \\ \boldsymbol{v}_{p_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix} \widehat{\boldsymbol{p}}_c \qquad \text{with} \qquad \boldsymbol{D}\left(\begin{bmatrix} \widehat{\boldsymbol{p}}_1 \\ \widehat{\boldsymbol{p}}_2 \end{bmatrix}\right) = \sigma_c^2 \begin{bmatrix} \boldsymbol{\Sigma}_1 & \boldsymbol{\emptyset} \\ \boldsymbol{\emptyset} & \boldsymbol{\Sigma}_2 \end{bmatrix}.$$

$$\Big[\underbrace{\boldsymbol{A}_1^T \boldsymbol{P}_1 \boldsymbol{A}_1}_{\boldsymbol{N_1}} + \underbrace{\boldsymbol{A}_2^T \boldsymbol{P}_2 \boldsymbol{A}_2}_{\boldsymbol{N_2}} \Big] \widehat{\boldsymbol{p}}_c = \Big[\underbrace{\boldsymbol{A}_1^T \boldsymbol{P}_1 \boldsymbol{y}_1}_{\boldsymbol{b_1}} + \underbrace{\boldsymbol{A}_2^T \boldsymbol{P}_2 \boldsymbol{y}_2}_{\boldsymbol{b_2}} \Big]$$

$$\Omega_c = \sum_{i=1}^{m} \boldsymbol{y}_i^T \boldsymbol{P}_i\, \boldsymbol{y}_i - \sum_{i=1}^{m} \boldsymbol{y}_i^T \boldsymbol{P}_i \boldsymbol{A}_i \widehat{\boldsymbol{p}}_c$$

$$\widehat{\sigma}_c^2 = \frac{1}{f_c} \left(\sum_{i=1}^{m} \boldsymbol{y}_i^T \boldsymbol{P}_i\, \boldsymbol{y}_i - \sum_{i=1}^{m} \boldsymbol{y}_i^T \boldsymbol{P}_i \boldsymbol{A}_i \widehat{\boldsymbol{p}}_c \right)$$

对称正定矩阵Cholesky分解方法

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \cdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & l_{22} & \cdots & l_{n2} \\ & & \ddots & \\ & & & l_{nn} \end{bmatrix}$$

STEP1 $l_{11} = \sqrt{a_{11}}$

STEP2 $l_{i1} = \dfrac{a_{i1}}{l_{11}}, i = 2, N$

STEP3 对 $j = 2, N$，做 STEP4~STEP5。

STEP4 $l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}$

STEP5 $l_{ij} = \dfrac{\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}\right)}{l_{jj}}, \quad i = j+1, N$

LDL分解

$$\boldsymbol{A} = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & l_{21} & \cdots & l_{n1} \\ & 1 & \cdots & l_{n2} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

$$d_1 = a_{11}$$

$$\left.\begin{aligned} g_{ij} &= a_{ij} - \sum_{k=1}^{j-1} g_{ik} l_{jk} \ (j = 1, \cdots, i-1) \\ l_{ij} &= \frac{g_{ij}}{d_j} \ (j = 1, \cdots, i-1) \\ d_i &= a_{ii} - \sum_{k=1}^{i-1} g_{ik} l_{ik} \end{aligned}\right\} i = 2, \cdots, n$$

LDL分解

```
subroutine chol_rf(A,L,d,P,N)
!---------------------------------subroutine  cor
!  Version    :  V1.0
!  Coded by   :  syz
!  Date       :  2019.04.15
!-----------------------------------------------
!  Purpose     :
!    不用开平方的Cholesky分解
!-- -----------------------------------------------
!  Input  parameters  :
!       1. A(N,N)--输入矩阵
!       2. N----矩阵维书
!  Output parameters  :
!       1.  L矩阵
!       2.  d---对角矩阵（用向量存储）
!       3.  P ---协方差矩阵（由于A已经为对称正定
!-----------------------------------------------
!  Copyrigt  :
!        Center for Astro-geodynamics
!        Shanghai Astronomical Observatory
!        Chinese Academy of Sciences
!-----------------------------------------------
implicit real*8(a-z)
integer::N
real*8::A(n,n),L(n,n),d(n),P(N,N)

!---------------------------------subroutine var
integer::i,j

real*8::  g(n,n)

!设置初值
L=0d0

d(1)=a(1,1)
```

```
do i=2,n
  do j=1,i-1
   !此层循环算g(i,j)
     tmp1=0d0

     do k=1,j-1
       tmp1=tmp1+g(i,k)*L(j,k)
     end do
     g(i,j)=a(i,j)-tmp1

  end do

  do j=1,i-1
  !此层循环算L(i,j)
    L(i,j)=g(i,j)/d(j)
  end do
    !以下算d(i)
  tmp1=0d0
  do k=1,i-1
   tmp1=tmp1+g(i,k)*L(i,k)
  end do

  d(i)=a(i,i)-tmp1
end do
!设置对角线元素
do i=1,N
   L(i,i)=1d0
end do
call covariance(L,D,P,N)
end subroutine chol_rf
```

由LDL分解计算协方差

```
subroutine covariance(L,D,P,N)
!-------------------------------------------------------
! Purpose : 2019-04-15 13:08                    (Created)
!   根据LDL'分解计算协方差矩阵
!-------------------------------------------------------
! Input Parameters    :
!    L   -----
!    D   ---- 对角线元素
!    N   ---- 矩阵维数
! Output Parameters   :
!-------------------------------------------------------
! Author       : Song Yezhi      <song.yz@foxmail.com>
! Copyrigt (C) : Shanghai Astronomical Observatory,CAS
!                (All rights reserved, 2019)
!-------------------------------------------------------
implicit none
integer     :: N
real*8      :: L(N,N),D(N),P(N,N)
!-------------------------------------------------------
integer     :: i
real*8      :: invL(N,N),invLT(N,N)
!-------------------------------------------------------
call inv_dtri(L,invL,N)
invLT = transpose(invL)
do i =1,N
    invLT(:,i)=invLT(:,i)/D(i)
end do
P = matmul(invLT,invL)
end subroutine covariance
```

LDL分解求解法方程

```
subroutine chol_eq(A,b,x,P,N)
!-----------------------------------su
!  Version    :  V1.0
!  Coded by   :  song.yz
!  Date       :  2019.04.15
!-----------------------------------
!  Purpose      : 对称正定方程解算（如
!-----------------------------------
!  Input  parameters  :
!       1.  A(N,N)对称正定系数矩阵
!       2.  b(N)右向量
!       3.  N--方程维数
!       4.
!  Output parameters  :
!       1.  x-计算结果
!       2.  P ----协方差 （由于A已经
!-----------------------------------
implicit real*8(a-z)

integer::N
real*8::A(N,N),b(n),x(n),P(N,N)

!-----------------------------------su
real*8::L(N,N),d(n)
integer::i,k
```

```
real*8::y(N)
call chol_rf(A,L,d,P,n)
y(1)=b(1)
do i=2,n
   tmp1=0d0
   do k=1,i-1
      tmp1=tmp1+L(i,k)*y(k)
   end do
   y(i)=b(i)-tmp1
end do
x(n)=y(n)/d(n)
do i=n-1,1,-1
  tmp1=0d0

  do k=i+1,n
  tmp1=tmp1+L(k,i)*x(k)
  end do

  x(i)=y(i)/d(i)-tmp1
end do

end subroutine chol_eq
```

下三角矩阵回代

```
subroutine inv_dtri(L,invL,N)
!-----------------------------------------------
! Purpose : 2019-04-15 10:45                  (C
!      计算下三角矩阵的逆矩阵，回带过程,无开平方
!
!-----------------------------------------------
! Input Parameters     :
!      L ----- 下三角矩阵
!      N ----- 矩阵维数
!
! Output Parameters    :
!      invL ------ 下三角的逆矩阵
!
!-----------------------------------------------
! Author        : Song Yezhi      <song.yz@foxma
! Copyrigt (C) : Shanghai Astronomical Observat
!                (All rights reserved, 2019)
!-----------------------------------------------

implicit none

integer    ::    N
real*8     ::    L(N,N),invL(N,N)

!-----------------------------------------------
integer    ::    i , j ,k
real*8     ::    xsum
!-----------------------------------------------


invL = 0d0

invL(1,1) = 1/L(1,1)
```

```
do i = 2, N

    invL(i,i)= 1/L(i,i)

    do j = 1,i-1

        xsum = 0d0

        do k = 1,i
            xsum = xsum + L(i,k)*invL(k,j)

        end do

        invL(i,j)= - xsum / L(i,i)

    end do

end do


end subroutine inv_dtri
```

法方程

```
subroutine  add_neq(ATA,ATb,oc,H,N)
!-----------------------------------subroutine  comment
!  Purpose    :   2018-09-05 13:33            (Created)
!              add a new observation to normal equations
!               (ATA+ H'H) x = ATb + H'oc
!  Changes    :
!       *.
!-------------------------------------------------------
!  Input Parameters     :
!       ATA  ----  initial normal matrix
!       Atb  ----  initial right vector
!       oc -----   o-c  for liner equation
!       H ------   obs vector
!       N -------  the dimention of the matrix
!  Output Parameters    :
!      ATA ------  updated normal matrix
!      ATb ------- updated right vector
!-------------------------------------------------------
!  Author        : Song Yezhi      <song.yz@foxmail.com>
!  Copyrigt (C)  : Shanghai Astronomical Observatory,CAS
!                 (All rights reserved, 2018)
!-------------------------------------------------------
implicit real*8(a-h,o-z)
real*8:: ATA(N,N),ATb(N) ,H(N)
do i = 1,N
     if (H(i)==0d0) cycle
     do j =1 ,N
       ATA(i,j) = ATA(i,j)+ H(i)*H(j)
     end do
     ATb(i)= ATb(i)+H(i)*oc
end do
end subroutine  add_neq
```

快速Givens变换

$$\overbrace{\begin{bmatrix} \overline{R} & \overline{\mathbf{b}} \\ H & \mathbf{y} \end{bmatrix}}^{n \quad 1} \begin{matrix} \}n \\ \}m \end{matrix} = \overbrace{\begin{bmatrix} \widetilde{R} \\ \widetilde{H} \end{bmatrix}}^{n+1} \begin{matrix} \}n \\ \}m. \end{matrix}$$

Sum = 0

$U_{ii} = 1 \qquad i = 1, \ldots, n$

1. Do $k = 1, \ldots, m$

$$\delta_k = 1$$

2. Do $i = 1, \ldots, n$

If $(h_{ki} = 0)$ Go to 2

$$\begin{aligned} d_i' &= d_i + \delta_k h_{ki}^2 \\ \overline{C} &= d_i / d_i' \\ \overline{S} &= \delta_k h_{ki} / d_i' \\ y_k' &= y_k - \tilde{b}_i h_{ki} \\ \tilde{b}_i &= \tilde{b}_i \overline{C} + y_k \overline{S} \\ y_k &= y_k' \\ \delta_k &= \delta_k \overline{C} \\ d_i &= d_i' \end{aligned}$$

3. Do $j = i + 1, \ldots, n$

$$\begin{aligned} h_{kj}' &= h_{kj} - U_{ij} h_{ki} \\ U_{ij} &= U_{ij} \overline{C} + h_{kj} \overline{S} \\ h_{kj} &= h_{kj}' \end{aligned}$$

Next j

Next i

$e_k = \sqrt{\delta_k} y_k$

Sum = Sum + e_k^2

Next k

Householder变换

如果给定向量 $\boldsymbol{x}, \boldsymbol{y}$，二者 2 范数相同，即 $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2$，则可以找到正交变换 $\boldsymbol{H}$，使 $\boldsymbol{Hx} = \boldsymbol{y}$。而变换矩阵很容易给出，即著名的 Householder 变换或称为镜像变换。取

$$\boldsymbol{u} = \frac{\boldsymbol{x} - \boldsymbol{y}}{\|\boldsymbol{x} - \boldsymbol{y}\|_2}$$

令

$$\boldsymbol{H} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T$$

即可。很容易证明 $\boldsymbol{H}$ 为正交矩阵，其变换保持向量 2 范数不变。

对于单个向量 $\boldsymbol{x}$，欲用镜像变换使之成为

$$\boldsymbol{Hx} = \boldsymbol{w} = \|\boldsymbol{x}\|\, \boldsymbol{e}_1, \boldsymbol{e}_1 = (1, 0, \cdots)^T$$

可以令

$$\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{x}$$

继而可以构造镜像变换矩阵

$$\boldsymbol{H} = \boldsymbol{I} - 2\frac{\boldsymbol{u}\boldsymbol{u}^T}{\|\boldsymbol{u}\|_2^2}$$

$\|\boldsymbol{u}\|_2^2$ 表示向量之 2 范数的平方。

Householder变换

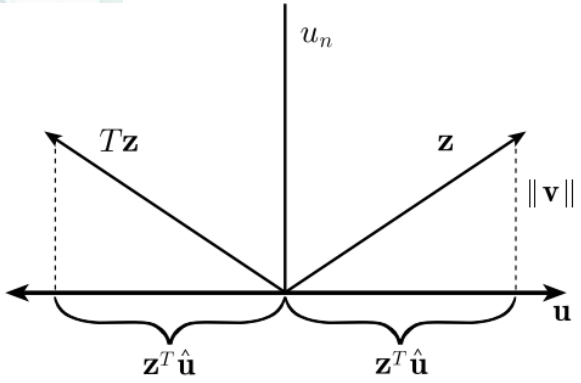
实施变换时为了让作为分母的 $\|\boldsymbol{u}\|_2^2$ 尽可能的大，从而有利于数值稳定，取

$$k = -\mathrm{sgn}(x_1)\|\boldsymbol{x}\|_2, \mathrm{sgn}(x_1) = \begin{cases} 1, x_1 \geqslant 0 \\ -1, x_1 < 0 \end{cases}$$

$$\boldsymbol{u} = (x_1 + \mathrm{sgn}(x_1)\|\boldsymbol{x}\|_2, x_2, \cdots, x_m)^T$$

意义很明了，及当 $\boldsymbol{x}$ 的第一个分量大于等于 0 时候，$\boldsymbol{u}$ 的第一个分量取 x_1 与 $\|\boldsymbol{x}\|_2$ 的和，当 $\boldsymbol{x}$ 的第一个分量小于 0 时候，$\boldsymbol{u}$ 的第一个分量取 $x_1 - \|\boldsymbol{x}\|_2$。如果不这样做有可能会损失有效位数。

对于矩阵 $\boldsymbol{A}$ 做 QR 分解，即连续使用 Houlseholder 变换。如第一次使用正交变换后，使之成为如下形式：



$$\boldsymbol{H}_1\boldsymbol{A} = \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

Householder变换

第二次变换使之成为如下形式：

$$\boldsymbol{H}_2\boldsymbol{H}_1\boldsymbol{A} = \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix}$$

如此循环到矩阵的最后一列，至此变换后的矩阵已经成为上三角矩阵。而变换矩阵即 $\boldsymbol{H} = \cdots \boldsymbol{H_2}\boldsymbol{H_1}$。

由数值代数理论可知，如果 $\boldsymbol{R}$ 对角元素正负号都选正号或者负号，则 QR 分解是唯一的。

计算 $\boldsymbol{H}$ 时候，并不需要存储各次的变换结果，而是逐步矩阵累成而得，变换到矩阵的最后一列，新的矩阵已经是 $\boldsymbol{H}$。上三角阵也不需要再执行 $\boldsymbol{R} = \boldsymbol{H}^T\boldsymbol{A}$，当变换到最后一列时已经自动形成上三角阵，该矩阵即为 $\boldsymbol{R}$。

Housholder变换

```
void householder(const MAT &A,MAT &Q,MAT &R)
/*----------------------------------------------
   Versions and Changes :
    v1.0---- 2015-09-02
       householder 变换
       REF: 《C#科学计算讲义》
----------------------------------------------
   Input Parameters   :
        A---要分解的矩阵
   Output Parameters  :
        Q---正交矩阵
        R---分解后的上三角矩阵
   Notes :
----------------------------------------------
   Author      : 宋叶志           <song.yz@foxmai
   Copyrigt(C) : Shanghai Astronomical Observato
                 (All rights reserved)
----------------------------------------------
{
   int M=A.size1();
   int N=A.size2();

   MAT H0(M,M);
   MAT H1(M,M);
   MAT H2(M,M);

   MAT A1(M,N);
   MAT A2(M,N);

   VEC u(M);

   int i,j;

   matcopy( A1, A);
   //copy matrix A to A1
```

```
for(i=0;i<M;i++)
 for(j=0;j<M;j++)
  H1(i,j)=0.0;
for(i=0;i<M;i++)
  H1(i,i)=1.0;


int k;
for(k=0;k<N;k++)
// k is index of all col
{
  //set the H to eye matrix
  for(i=0;i<M;i++)
   for(j=0;j<M;j++)
    H0(i,j)=0.0;

  for(j=0;j<M;j++)
   H0(j,j)=1.0;

  double s=0.0;
  for(i=k;i<M;i++)
   s=s+A1(i,k)*A1(i,k);

  s=sqrt(s);

  for(i=0;i<M;i++)
   u(i)=0.0;

  if (A(k,k)>=0.0)
     u(k)=A1(k,k)+s;
  else
     u(k)=A1(k,k)-s;
```

Householder变换

```
for(i=k+1;i<M;i++)
   u(i)=A1(i,k);

double du;
du=vecdot(u,u);

for(i=k;i<M;i++)
 for(j=k;j<M;j++)
  {

   H0(i,j)=-2.0*u(i)*u(j)/du;
   if(i==j)
       H0(i,j)=1.0+H0(i,j);

  }

matmul(A2,H0,A1);
matcopy(A1,A2);
```

```
    //在 C#科学计算讲义中（song.yz）
    //直接 matmul(H1,H1,H0)
    //这里不行，因为 matmul中使用 cc
    matmul(Q,H1,H0);
    matcopy(H1,Q);

  }

  matcopy(Q,H1);
  matcopy(R,A1);


//------set the lower part of matrix
//  in fact ,it is close to zero afte
//  that means it is no necessary to
  for(i=0;i<N;i++)
   for(j=i+1;j<M;j++)
     R(j,i)=0.0;

}
```

修正的Gram-Schimdt 正交化方法

设 $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$ 是 p 维向量子空间 W 的任意一组基，则子空间 W 的标准正交基 $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}$ 可以通过 Gram-Schimdt 正交化构造，这个方法是大家都很熟悉的，即

$$\boldsymbol{p}_1 = \boldsymbol{x}_1, \boldsymbol{u}_1 = \frac{\boldsymbol{p}_1}{\|\boldsymbol{p}\|} = \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}$$

$$\boldsymbol{p}_k = \boldsymbol{x}_k - \sum_{i=1}^{k=1} \left(\boldsymbol{v}_i^H \boldsymbol{x}_k\right) \boldsymbol{u}_i, \boldsymbol{u}_k = \frac{\boldsymbol{p}_k}{\|\boldsymbol{p}_k\|}$$

对于超定的线性方程系数矩阵的 QR 分解，可以通过 Gram-Schimdt 正交化方法来实现，然而采用 Gram-Schimdt 正交化方法求解列正交矩阵 $\boldsymbol{Q}$ 时，舍入误差较大，这在求解最小二乘法时候，有时会不稳定。针对 Gram-Schimdt 正交化的缺点，下面给出修正的 Gram-Schimdt 正交化算法。

修正的Gram-Schimdt 正交化方法

对于 n 个向量 $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ 构造标准正交基 $\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_n$ 方法如下：

$$R_{11} = \|\boldsymbol{a}_1\|$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{q}_1}{R_{11}}$$

对于 $k = 2, \cdots, n$

$$R_{jk} = \boldsymbol{q}_j^H \boldsymbol{a}_k, j = 1, \cdots, k-1$$

$$R_{kk} = \left\| \boldsymbol{a}_k - \sum_{j=1}^{k-1} \boldsymbol{q} R_{jk} \right\|$$

$$\boldsymbol{q} = \frac{\boldsymbol{a}_k - \sum_{j=1}^{k-1} \boldsymbol{q} R_{jk}}{R_{kk}}$$

MGS正交化过程

```
void MGS(const MAT &A, MAT &Q, MAT &R)
 /*-----------------------------------------
 Author       : 宋叶志
 Date         : 2015-7-28
 -----------------------------------------
 Desciption   :  修正的Gram-Schmidt正交化方法
  *
 Post Script :
  *           分解结果为 Q是列正交的基向量，R
  *           注意分解结果与Householder变换是
 parameters   :
  *      A----原矩阵
  *      Q----正交基组成的向量空间
  *      R----方阵
 Ref:
        1. scientific computation in C#
        2. scientific computation in Fortran
           song.yz   Tsinghua Univ. Press ,20
 -----------------------------------------*/
{
    //获取A矩阵维数
    int M, N;
    M = A.size1();
    N = A.size2();

    VEC vtmp(M);

    double s=0.0;

    int i, j;
```

```
for (i = 0; i < M; i++)
    s = s + A(i, 0) * A(i, 0);

R(0, 0) = sqrt(s);

for (i = 0; i < M; i++)
    Q(i, 0) = A(i, 0) / R(0, 0);


for (int k = 1; k < N; k++)
{
    for (j = 0; j <= k - 1; j++)
    {
        s=0;
        for (int p = 0; p < M; p++)
            s = s + Q(p, j) * A(p, k);
        R(j, k) = s;
    }

    for (i = 0; i < M; i++)
        vtmp(i) = A(i, k);

    for (j = 0; j <= k - 1; j++)
        for (i = 0; i < M; i++)
            vtmp(i) = vtmp(i) - Q(i, j) * R(j, k);

    R(k,k)=sqrt(vecdot(vtmp,vtmp));

    for (i = 0; i < M; i++)
        Q(i, k) = vtmp(i) / R(k, k);

}
```

扩展卡尔曼滤波

Given: P_{k-1}, $\hat{\mathbf{X}}_{k-1}$ and $\mathbf{Y}_k$, R_k.

(1) Integrate from t_{k-1} to t_k,

$$\dot{\mathbf{X}}^* = F(\mathbf{X}^*, t), \qquad \mathbf{X}^*(t_{k-1}) = \hat{\mathbf{X}}_{k-1}$$
$$\dot{\Phi}(t, t_{k-1}) = A(t)\Phi(t, t_{k-1}), \; \Phi(t_{k-1}, t_{k-1}) = I.$$

(2) Compute

$$\overline{P}_k = \Phi(t_k, t_{k-1}) P_{k-1} \Phi^T(t_k, t_{k-1})$$
$$\mathbf{y}_k = \mathbf{Y}_k - G(\mathbf{X}_k^*, t_k)$$
$$\widetilde{H}_k = \partial G(\mathbf{X}_k^*, t_k)/\partial \mathbf{X}_k.$$

(3) Compute

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$
$$\hat{\mathbf{X}}_k = \mathbf{X}_k^* + K_k \mathbf{y}_k$$
$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k.$$

(4) Replace k with $k+1$ and return to (1).

cmatlib矩阵类库定义（C++）

```
class VEC
/*------------------------------------------------------------
   Versions and Changes :
      v1.0----2015-7-18
          vector class
          下标索引算符为()
------------------------------------------------------------
   Public Paras:
           int size1 ---------  rows  of matrix
           int size2 ---------  cols  of matri
   Funcitons :
          VEC (M)  ---- creat a vector with lenght of M
          operator() --- access the component
------------------------------------------------------------
   Author      : 宋叶志  <song.yz@foxmail.com>
   Copyrigt(C) : Shanghai Astronomical Observatory, CAS
                 (All rights reserved)              2015
------------------------------------------------------------*/
{
   private:
      double *x ;
      int M1;
   public:
      // Constructors
      VEC (int M)
      //creat a vector
      {
        M1=M;
        x= new double [M] ;
        for(int i=0;i<M;i++)
          x[i]=0.0;
      }
      // Destructor
      ~ VEC ()
      { delete [] x; }

      double& operator() (int i)
      // Component access
      { return x[i]; }
      double  operator () (int i) const { return x[i]; };
      int size() const { return M1; };
      void  output(int width,int precision);
      //output to the screen
      void  setv(double value);
      //set the elements of the vector to a double value
      //+,-,*,/ for the same index of two vectors
      //which means that the size of the vectors must be the same
      friend VEC operator + (const VEC& V1, const VEC& V2); //V1+V2
      friend VEC operator - (const VEC& V1, const VEC& V2); //V1-V2
      friend VEC operator * (const VEC& V1, const VEC& V2); //V1.*V2,
      friend VEC operator / (const VEC& V1, const VEC& V2); //V1./V2,
      //V1./V2
      //+,-,*,/ for the vector and a scala
      // the operation works on every elements of the vector
      friend VEC operator + (const VEC& V1, double a);     //V1+a
      friend VEC operator - (const VEC& V1, double a);     //V1-a
      friend VEC operator * (const VEC& V1, double a);     //V1*a
      friend VEC operator / (const VEC& V1, double a);     //V1/a
} ;
```

cmatlib矩阵类库定义

```
class MAT
/*----------------------------------------------------------
   Versions and Changes :
      v1.0----2015-7-18
         matrix class
          在数组基础上构造,下标算符重载()
----------------------------------------------------------
   Public Paras:
           int size1 ---------  rows  of matrix
           int size2 ---------  cols  of matrix
   Functions   :
           MAT(M,N)  ---- creat a vector with the size of (M,N)
           operator() ---  caccess the component
----------------------------------------------------------
   Author      : 宋叶志  <song.yz@foxmail.com>
   Copyrigt(C) : Shanghai Astronomical Observatory, CAS
                 (All rights reserved)            2015
----------------------------------------------------------*/
{
   private:
     double ** A;
     int M1,N1;
   public:
     // Constructors
     MAT(int M,int N)
     {
        M1=M;
        N1=N;

        int i,j;

        A= new double *[M];
        for(i=0;i<M;i++)
          A[i]=new double [N] ;


        // set the initial value to zero
        for (i=0;i<M;i++)
         for(j=0;j<N;j++)
            A[i][j]=0.0;
     }
```

```
     // Destructor
     ~MAT()
     {
        int i;
        for(i=0;i<M1;i++)
          delete[] A[i];
        delete[] A;
     }

   //Component access
     double   operator() (int i,int j) const  { return A[i][j]; };
     double & operator() (int i,int j)        { return A[i][j]; };


     int size1() const { return M1; };
     int size2() const { return N1; };

     void output(int width,int precision);
     //output to the screen
     void setv(double value);
     //set the elements of the matrix to a double value

     //+,-,*,/ for the same index of two matrixs
     //which means that the size of the matrixs must be the same
     friend MAT operator + (const MAT& A1, const MAT& A2);   //A1+A2
     friend MAT operator - (const MAT& A1, const MAT& A2);   //A1-A2
     friend MAT operator * (const MAT& A1, const MAT& A2);   //A1.*A2
     friend MAT operator / (const MAT& A1, const MAT& A2);   //A1./A2

     //+,-,*,/ for the matrix and a scala
     // the operation works on every elements of the matrix
     friend MAT operator + (const MAT& A1, double a);      //A1+a
     friend MAT operator - (const MAT& A1, double a);      //A1-a
     friend MAT operator * (const MAT& A1, double a);      //A1*a
     friend MAT operator / (const MAT& A1, double a);      //A1/a

};
```

cmatlib向量运算与矩阵分解

```
//--------------基本向量运算
void veccopy(VEC &b,const VEC &a);
//按值复制一个向量
double vecdot(const VEC &v1,const VEC & v2);
//向量内积
double norm(const VEC &v);
//向量2范数
//-----------------基本矩阵运算
void matcopy(MAT &B,const MAT &A);
//矩阵复制
void transpose(MAT &AT,const MAT &A);
//矩阵转置
void matmul(VEC &b,const MAT &A,const VEC &x);
// 矩阵乘以向量
void matmul(MAT &C,const MAT &A,const MAT &B);
//矩阵乘以矩阵
```

```
//------------------矩阵分解与求逆
void LDL(const MAT &A,MAT &L,VEC &D);
// LDL 分解
void MGS(const MAT &A, MAT &Q, MAT &R);
//修正的Gram-Schmidt正交化方法
void householder(const MAT &A,MAT &Q,MAT &R);
//hoseholder正交变换
void invlowtri(const MAT &R,MAT &S);
//inverse of lower triangula matrix
void invuptri(const MAT &U,MAT &R);
//inverse of upper triangula matrix
void invmat(const MAT &A,MAT &invA);
//对称正定矩阵逆矩阵
void inv(const MAT &A,MAT &iA);
//一般矩阵的逆矩阵 采用MGS分解方法
```

cmatlib求解线性代数方程

```
//--------------------线性代数方程
void LS_LDL(VEC &x,const MAT &A,const VEC &b);
//基于不开平方的cholesky分解计算最小二乘问题， A为对称正定矩阵
//void LS_hous(VEC &x,const MAT &A,const VEC &b);
//基于Householder变换求解最新二乘问题或适定问题
void LS_MGS( VEC &x,const MAT &A,const VEC &b);
//通过修正的Gram-Schimdt正交化求解最小二乘问题或适定问题
void LS_hous( VEC &x,const MAT &A,const VEC &b);
//least square solution or general linear equation
void uptri(const MAT &A,const VEC &b,VEC &x);
//上三角矩阵方程计算
void lowtri(const MAT &A, const VEC &b, VEC &x);
//下三角矩阵方程计算
void downtri(const MAT &A, const VEC &b, VEC &x);
//-------------------插值算法
void lagrange(const VEC &x,const VEC &y,const VEC &xx,VEC &yy);
//lagrange interp
void rot_mat(MAT &Rmat,double angle,char axID);
//rotation matrix
double robustweight(double v,double sigma);
```


Q&A!